Models of growth with capital contributing to utility

Petr Duczynski*

Department of Economics and Management,
Faculty of Informatics and Management,
University of Hradec Králové, Rokitanského 62,
500 03 Hradec Králové, Czech Republic

Abstract

The paper examines several growth models in which capital enters into the utility function. The inclusion of capital in utility seems especially attractive for human capital. The presence of capital in utility can increase the steady-state growth rate in endogenous growth models and decrease the speed of convergence in exogenous growth models. An extended AK model with human capital in utility results in an empirically plausible initial imbalance effect between human and physical capital.

Keywords: Endogenous and exogenous growth; Human capital; Physical capital; Utility

JEL classification: O41

*E-mail: petr.duczynski@uhk.cz. This research was partially supported by the Grant Agency of the Czech Republic, grant number 402/04/0642.

1 Introduction

This paper considers seven models of growth in which the instantaneous utility depends on consumption and on the level of the capital stock. The inclusion of capital in utility constitutes an innovative contribution of this paper. Standard models of growth assume that utility depends only on consumption (see, for example, Barro and Sala-i-Martin, 1995). In real business cycle models, utility depends on consumption and leisure (see, among many others, Kydland and Prescott, 1982, and King, Plosser, and Rebelo, 1988). Although these approximations can be satisfactory in some respects, they do not provide a full description of reality. One possibility of a better description of reality is the inclusion of human capital in utility.[1] In the real world, utility definitely depends on human capital in the form of health. This relevant consideration has been paid little attention in the growth literature. Yet it seems that the importance of health for utility is comparable to the importance of consumption.

Human capital in the form of education can also contribute to utility. It is plausible that education increases the quality of life even if one abstracts from the effect of education on income.[2] For example, a lack of literacy prevents people from reading books, thus decreasing the productivity of leisure in utility. Some forms of education (such as in the field of history or philosophy) are not directly productive in the real economy, but they contribute to a better orientation in the world. Last but not least, one can consider the past experience of countries with low rates of return on education (such as Eastern European countries during the period of central planning).[3] A relatively high demand for higher schooling in these countries probably reflects that education contributes to utility. According to the Soviet statistical yearbook (*Narodnoye chozyaystvo SSSR*, 1983), the average monthly wages in the U.S.S.R. in 1980 were 127 roubles in the sector of healthcare, physical training, and social security, 136 roubles in the sector of national education, 111 roubles in the sector of culture, 135 roubles in the sector

[1] Ortigueira and Santos (1997) examine endogenous growth models in which human capital increases the productivity of leisure in utility.

[2] The educational attainment index is (together with the life expectancy index and the adjusted real GDP per capita index) used in the construction of the human development index in the *Human Development Report* of the United Nations Development Programme.

[3] Filer, Jurajda, and Plánovský (1999) examine Czech and Slovak data and show that returns on education substantially increased during transition towards a market economy.

of art, and 180 roubles in the sector of science. In comparison, the average wage of industrial workers (for which the education is plausibly lower; engineers are excluded) was 186 roubles. The observed negative rates of return on education can be consistent with a rational behavior of individuals if human capital is a source of utility.

Even for physical capital one can find some justification for including it in utility.[4] Long-run growth models typically work in a deterministic framework, thus neglecting the aspects of uncertainty. In the real world with uncertainty, physical capital can be accumulated in order to secure a reasonable level of consumption in bad times (this precautionary motive of capital accumulation applies if the third derivative of utility with respect to consumption is positive). This phenomenon can be reflected in a deterministic framework if capital directly contributes to utility. Rich people enjoy the fact that their physical capital provides them with some level of security.[5]

Alternatively, physical capital can be valued because it provides individuals with an option to make purchases across a variety of goods. The ability to make a choice can be a source of utility even if no purchases are actually realized. Physical capital extends the degree of freedom of individuals. A similar argument would also apply for human capital: Educated people can make a choice over a larger variety of jobs.

Another channel of the effects of capital on utility is via future expected consumption. It is plausible that the present utility depends not only on the present consumption, but also on the future expected consumption. Rich people may be happy because they know that their future consumption will be high. If I knew that I would win a large sum of money in a lottery in two years, my present utility would be high even though my present consumption was low (say because of tight borrowing constraints).

The first model is a natural extension of the AK model (for the AK model, see Barro and Sala-i-Martin, 1995, Chapter 4). Capital can be thought to be a composite of human and physical capital. In this model, the presence of capital in utility increases the growth rate of the economy. Similarly to the basic AK

[4] The role of money in utility (Sidrauski, 1967) could also be justified along these lines.

[5] This paragraph's contents can be roughly expressed in a way that "securities provide security."

model, there are no transitional dynamics.

The second model is a straightforward extension of the Ramsey model. The formula for the speed of convergence of the log-linearized model can be found analytically. The presence of capital in utility leads to a lower convergence coefficient if utility is close to an additively logarithmic case.

Similarly to the first model, the third model is an extension of the AK model. This model introduces an additional capital stock (human capital) that is assumed to enter utility. In the model's solution, physical capital, human capital, output, and consumption grow at the same rate; there are no transitional dynamics. An important feature of the model is an imbalance effect that arises if the initial ratio of human to physical capital differs from its desired value. The model predicts an immediate adjustment of the ratio of human to physical capital to its steady-state value. In a more realistic framework with irreversibility restrictions and adjustment costs, the adjustment would be gradual, and the growth of output would depend positively on the initial ratio of human to physical capital. This is consistent with empirical observations.

The fourth model is a standard one-sector two-capital growth model that is again extended by human capital contributing to utility. This model uses the Cobb-Douglas production function and is more complicated than the previous model. The analysis is focused on the steady-state solution. The presence of human capital in utility tends to increase the steady-state growth rate.

The fifth model is an extension of the Uzawa-Lucas model (see Uzawa, 1965, and Lucas, 1988) in which human capital enters into the utility function. The analysis is focused on the steady-state solution, which describes the evolution of the economy in the long run. It is possible to find a closed-form solution for the steady-state growth rate, and it turns out that this growth rate increases if human capital is included in utility.

The sixth model extends the Uzawa-Lucas model by adding physical capital to utility. There is no change in the steady-state growth rate of the economy.

The seventh model is a two-sector model with three types of capital. The type of capital that indicates access to education (called the educational capital) enters into utility. The inclusion of the economy-wide level of this capital in utility increases the steady-state growth rate of the economy in the planner's solution. The decentralized outcome is suboptimal and the social optimum can be achieved if the educational capital is subsidized at the expense of lump-sum taxes. The social optimum cannot be achieved if the subsidies to the educational capital are financed by taxes on output.

Social planner's solutions are considered throughout the paper. Because of the absence of externalities, these solutions coincide with decentralized solutions (with the exception of the seventh model). There is no population growth and no technical change; individual variables can be interpreted as per capita variables.

2 One type of capital

2.1 An extended AK model

Let the production depend linearly on capital:

$$Y = AK, \tag{1}$$

where Y is output, A is a technological parameter, and K is capital (plausibly a composite of physical and human capital). The problem is to maximize

$$\max_C \int_0^\infty e^{-\rho t}(\ln C + \gamma \ln K)dt$$

subject to

$$\dot{K} = AK - \delta K - C, \tag{2}$$

where C is consumption, ρ is the rate of time preference, γ is a positive parameter, and δ is the depreciation rate for capital. The present-value Hamiltonian for this problem is

$$\mathcal{H} = e^{-\rho t}(\ln C + \gamma \ln K) + \lambda(AK - \delta K - C), \tag{3}$$

where λ is the marginal shadow value of K. The first-order conditions are

$$e^{-\rho t}/C = \lambda, \tag{4}$$

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial K} = -\gamma e^{-\rho t}/K + \lambda(\delta - A). \tag{5}$$

The transversality condition is

$$\lim_{t\to\infty} \lambda K = 0. \tag{6}$$

Table 1: The dependence of the growth rate on the weight of capital in utility in the extended AK model for $A = 0.1$, $\delta = 0.05$, and $\rho = 0.03$.

γ	0	0.2	0.4	0.6	0.8	1	2	3	∞
g (%)	2.0	2.5	2.9	3.1	3.3	3.5	4.0	4.3	5.0

In a steady state, all variables are required to grow at constant rates. From the equation of motion for capital it follows that the ratio of consumption to capital is constant in a steady state; hence capital and consumption grow at the same rate. From the first-order conditions it follows that

$$\frac{\dot{\lambda}}{\lambda} = -\gamma\frac{C}{K} + \delta - A, \tag{7}$$

$$\frac{\dot{C}}{C} = A - \delta + \gamma\frac{C}{K} - \rho. \tag{8}$$

The equations of motion for capital and consumption imply that if the ratio of consumption to capital differs from its steady-state value, it further deviates from this value. Thus the steady-state solution is the only stable solution; there are no transitional dynamics. The equality of the growth rates for consumption and capital indicates that

$$\frac{C}{K} = \frac{\rho}{1+\gamma}, \tag{9}$$

$$g = A - \delta - \frac{\rho}{1+\gamma}, \tag{10}$$

where g is the common growth rate of consumption, capital, and output. Thus the presence of capital in utility (the parameter γ) tends to increase the growth rate of the economy. Appendix A shows that this property is still satisfied if utility is generalized to a constant elasticity of intertemporal substitution case.

Table 1 shows the dependence of g on γ for $A = 0.1$, $\delta = 0.05$, and $\rho = 0.03$. Even for small values of γ the effect on growth is relatively important.

2.2 An extended Ramsey model

The problem is to maximize

$$\max_C \int_0^\infty e^{-\rho t}(\ln C + \gamma \ln K)dt$$

subject to

$$\dot{K} = F(K) - \delta K - C, \tag{11}$$

where $F(K)$ is an increasing and strictly concave function. K is again a composite of human and physical capital. The present-value Hamiltonian is

$$\mathcal{H} = e^{-\rho t}(\ln C + \gamma \ln K) + \lambda[F(K) - \delta K - C], \tag{12}$$

where λ is the marginal shadow value of K. The first-order conditions are

$$e^{-\rho t}/C = \lambda, \tag{13}$$

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial K} = -\gamma e^{-\rho t}/K + \lambda[\delta - F'(K)]. \tag{14}$$

The transversality condition is

$$\lim_{t \to \infty} \lambda K = 0. \tag{15}$$

Consumption and capital are constant in the steady state. The steady-state values, C^* and K^*, satisfy

$$F(K^*)/K^* = \delta + C^*/K^*, \tag{16}$$

$$0 = -\rho + F'(K^*) - \delta + \gamma[F(K^*)/K^* - \delta]. \tag{17}$$

Let us consider the Cobb-Douglas production function:

$$F(K) = AK^\alpha \tag{18}$$

For this function it holds that

$$F(K^*)/K^* = \frac{\rho + \delta + \gamma\delta}{\alpha + \gamma}, \tag{19}$$

$$F'(K^*) = \frac{\alpha(\rho + \delta + \gamma\delta)}{\alpha + \gamma}, \tag{20}$$

$$F''(K^*)K^* = \frac{\alpha(\alpha - 1)(\rho + \delta + \gamma\delta)}{\alpha + \gamma}. \tag{21}$$

The equations of motion for capital and consumption can be log-linearized around the steady state. In this approximation, the equations are:

$$\frac{d \ln K}{dt} = [F'(K^*) - \delta] \ln(K/K^*) - [F(K^*)/K^* - \delta] \ln(C/C^*), \tag{22}$$

$$\frac{d\ln C}{dt} = \{F''(K^*)K^* - \gamma[F(K^*)/K^* - \delta]\}\ln(K/K^*) + \gamma[F(K^*)/K^* - \delta]\ln(C/C^*). \tag{23}$$

The solution to this system of two differential equations with constant coefficients consists in finding the negative eigenvalue, $-\beta$ (the terms with the positive eigenvalue must be excluded for the transversality condition to be satisfied), of the Jacobian matrix. Then the solution for capital is given by

$$\ln(K/K^*) = \ln[K(0)/K^*]e^{-\beta t}. \tag{24}$$

Because the production function is Cobb-Douglas, a similar solution applies for output:

$$\ln(Y/Y^*) = \ln[Y(0)/Y^*]e^{-\beta t}, \tag{25}$$

where Y^* is the steady-state value of output. The convergence coefficient β satisfies

$$\beta = \frac{\sqrt{\rho^2 + 4(1-\alpha)(\rho+\delta+\gamma\delta)(\rho+\delta-\alpha\delta)/(\alpha+\gamma)} - \rho}{2}. \tag{26}$$

From this it follows that β decreases with γ, i.e., the presence of capital in utility slows down the speed of convergence. It should also be noted that the extreme case with $\gamma = 0$ coincides with the formula for the speed of convergence presented in Barro and Sala-i-Martin (1995, Chapter 2) (if one puts the technical change and population growth equal to zero[6] and the elasticity of intertemporal substitution of consumption equal to one).

To see quantitatively how β depends on γ, I considered the specification $\rho = 0.03$ and $\delta = 0.05$. Table 2 shows the dependence if $\alpha = 0.75$, which is a realistic capital share if capital is viewed broadly (consisting of physical and human components). Table 3 shows the same dependence if $\alpha = 0.30$, which is a capital share corresponding to narrowly viewed capital (physical capital only).

Appendix B shows that the decreasing dependence of β on the weight of capital in utility is not precisely satisfied if utility is generalized to a constant elasticity of intertemporal substitution case. Nevertheless, β still depends negatively on the weight of capital if this weight is not very large and if the inverse elasticity of intertemporal substitution is reasonably low (such as lower than 2).

[6]If we introduced population growth (at the rate of n) and labor-augmenting technical change (at the rate of x), then we would obtain exactly the same problem with δ replaced with $\delta + n + x$ and ρ replaced with $\rho - n$. The speed of convergence would still depend negatively on γ.

Table 2: The dependence of the convergence coefficient on the weight of capital in utility for $\rho = 0.03$, $\delta = 0.05$, and $\alpha = 0.75$.

γ	0	0.2	0.4	0.6	0.8	1	2	3	∞
β (%)	2.2	2.0	1.9	1.8	1.7	1.7	1.5	1.5	1.3

Table 3: The dependence of the convergence coefficient on the weight of capital in utility for $\rho = 0.03$, $\delta = 0.05$, and $\alpha = 0.30$.

γ	0	0.2	0.4	0.6	0.8	1	2	3	∞
β (%)	9.6	7.7	6.7	6.1	5.7	5.4	4.7	4.3	3.5

3 Two types of capital

3.1 One sector with an AK production

Let the production depend linearly on physical capital:

$$Y = AK, \tag{27}$$

where Y is output, A is a technological parameter, and K is physical capital. The problem is to maximize

$$\max_{C, I_H} \int_0^\infty e^{-\rho t}(\ln C + \gamma \ln H) dt$$

subject to

$$\dot{K} = AK - \delta K - C - I_H, \tag{28}$$

$$\dot{H} = I_H - \delta H, \tag{29}$$

where H is human capital and I_H is gross investment in human capital.[7] The present-value Hamiltonian is

$$\mathcal{H} = e^{-\rho t}(\ln C + \gamma \ln H) + \lambda_K(AK - \delta K - C - I_H) + \lambda_H(I_H - \delta H),$$

[7] In this model, K may correspond to a composite of physical capital and the part of human capital that is productive (such as the knowledge of business administration). H then corresponds to an unproductive part of human capital (such as the knowledge of philosophy, philology, or history) that enhances utility.

where λ_K and λ_H are the marginal shadow values of physical and human capital, respectively. The first-order conditions are:

$$e^{-\rho t}/C = \lambda_K, \tag{30}$$

$$\lambda_K = \lambda_H, \tag{31}$$

$$\dot{\lambda}_K = -\frac{\partial \mathcal{H}}{\partial K} = \lambda_K(\delta - A), \tag{32}$$

$$\dot{\lambda}_H = -\frac{\partial \mathcal{H}}{\partial H} = -e^{-\rho t}\gamma/H + \delta\lambda_H. \tag{33}$$

The transversality conditions are

$$\lim_{t\to\infty} K\lambda_K = 0, \tag{34}$$

$$\lim_{t\to\infty} H\lambda_H = 0. \tag{35}$$

From the first-order conditions it follows that

$$\frac{\dot{C}}{C} = A - \delta - \rho, \tag{36}$$

$$\frac{C\gamma}{H} = A, \tag{37}$$

$$\frac{\dot{H}}{H} = A - \delta - \rho, \tag{38}$$

$$\frac{\dot{K}}{K} = A - \delta - \left(\frac{A}{\gamma} + A - \rho\right)\frac{H}{K}. \tag{39}$$

In the steady state, the ratio of human to physical capital is constant and it satisfies

$$\left(\frac{A}{\gamma} + A - \rho\right)\frac{H}{K} = \rho. \tag{40}$$

If the ratio of human to physical capital is higher (lower, respectively) than the steady-state ratio, the growth rate of physical capital is lower (higher, respectively) than the growth rate of human capital. This cannot be an equilibrium solution converging to the steady state. Thus the only plausible solution is the steady-state solution; there are no transitional dynamics.

If the initial ratio of human to physical capital differs from its steady-state ratio, there is an immediate adjustment of human and physical capital. If the initial ratio of human to physical capital exceeds its steady-state value, human capital is immediately transformed into physical capital, and output increases. If, on the other hand, the initial ratio of human to physical capital is below its steady-state value, physical capital is immediately transformed into human capital, and output decreases. These transformations are immediate because there are no irreversibility restrictions or adjustment costs for investment. A more realistic framework would require the introduction of irreversibility restrictions and/or adjustment costs. In this case, the initial adjustment would be spread gradually over time, and the initial growth of output would be positively related to the initial ratio of human to physical capital; the model would exhibit an imbalance effect.

A positive dependence of the growth rate on the ratio of human to physical capital is empirically plausible. Following Barro (1991), numerous empirical studies have shown that for a given level of output, economic growth depends positively on human capital. Economies with high H/K ratios, such as West Germany and Japan after World War II, really tended to grow rapidly. On the other hand, there is some evidence that growth was not fast if the H/K ratio was low (see Hirshleifer, 1987, Chapters 1 and 2 for a discussion of the Black Death). Additionally, I provide evidence (Duczynski, 2003) that in a sample of 73 countries for which I have data, the growth of per capita output between 1960 and 1990 depended significantly positively on the initial ratio of human to physical capital.

An imbalance effect occurs in other growth models. Elsewhere (Duczynski, 2002) I show that the output growth rate in the log-linearized two-capital model with perfect capital mobility, large adjustment costs for human capital, and small adjustment costs for physical capital depends positively on the ratio of human to physical capital. Barro and Sala-i-Martin (1995, Chapter 5) examine the imbalance effects in one-sector and two-sector closed-economy models with two types of capital. In the one-sector endogenous growth model with irreversibility restrictions, the growth rate as a function of the H/K ratio is U-shaped: the growth rate decreases with H/K if H/K is small and increases with H/K if H/K is large. In the two-sector endogenous growth model (Uzawa-Lucas model), the growth rate of the output of goods tends to be also U-shaped as a function of H/K. However, the growth rate of broad output (the output of goods plus the value of gross investment in human capital in units of goods) depends positively on H/K for a broad range of H/K. In this model, the imbalance effect occurs as a result of the human-capital intensity of the educational sector. In comparison, the imbalance effect in the model presented in this subsection is influenced by diminishing marginal utility from human capital. If the ratio of human to physical capital is large initially, utility benefits from human capital are outweighted by production gains from physical capital; the economy invests intensively in physical capital and the growth of output is fast. On the other hand, if the economy is relatively human-capital scarce, it must invest some resources in unproductive components of human capital (see footnote 7), which slows down the output growth rate.[8]

3.2 One sector with a Cobb-Douglas production

The production of output is given by

$$Y = AK^{\alpha}H^{1-\alpha}. \tag{41}$$

In other respects, the model is similar to the previous model. A corresponding model with no human capital in utility is discussed in Barro and Sala-i-Martin (1995, Chapter 5). The present-value Hamiltonian is

$$\mathcal{H} = e^{-\rho t}(\ln C + \gamma \ln H) + \lambda_K(AK^{\alpha}H^{1-\alpha} - \delta K - C - I_H) + \lambda_H(I_H - \delta H).$$

The first-order conditions are

$$e^{-\rho t}/C = \lambda_K, \tag{42}$$

$$\lambda_K = \lambda_H, \tag{43}$$

$$\dot{\lambda}_K = -\frac{\partial \mathcal{H}}{\partial K} = \lambda_K[\delta - \alpha A(H/K)^{1-\alpha}], \tag{44}$$

[8] A problematic aspect of the present model may be that H does not enter into production. This assumption is abandoned in the following subsection. The following subsection examines a model in which the steady-state ratio of human to physical capital is above this ratio in a model with no human capital in utility. If the steady-state solution were the only solution, the given model would exhibit a more satisfactory initial imbalance effect than a corresponding model with no capital in utility (the initial output growth would depend positively on the initial H/K ratio over a larger range of the initial H/K ratio). Nevertheless, I am not able to prove that the steady-state solution is the only solution.

$$\dot{\lambda}_H = -\frac{\partial \mathcal{H}}{\partial H} = -\frac{e^{-\rho t}\gamma}{H} + \delta\lambda_H - \lambda_K(1-\alpha)A(K/H)^{\alpha}. \tag{45}$$

The first-order conditions lead to

$$\frac{\dot{C}}{C} = \alpha A(H/K)^{1-\alpha} - \delta - \rho, \tag{46}$$

$$\frac{C\gamma}{H} = \alpha A(H/K)^{1-\alpha} - (1-\alpha)A(H/K)^{-\alpha}, \tag{47}$$

$$\frac{\dot{H}}{H} = \frac{\dot{C}}{C} - \frac{d}{dt}\ln[\alpha A(H/K)^{1-\alpha} - (1-\alpha)A(H/K)^{-\alpha}], \tag{48}$$

$$\frac{\dot{K}}{K} = A(H/K)^{1-\alpha} - \delta - \frac{H}{K\gamma}[\alpha A(H/K)^{1-\alpha} - (1-\alpha)A(H/K)^{-\alpha}] - \frac{H}{K}\{\alpha A(H/K)^{1-\alpha} - \rho - \frac{d}{dt}\ln[\alpha A(H/K)^{1-\alpha} - (1-\alpha)A(H/K)^{-\alpha}]\}. \tag{49}$$

The model is greatly complicated; therefore, the analysis is focused only on the steady-state solution. In the steady state, H/K is constant, and the growth rates of H, K, Y, and C are the same. The equality of the growth rates of K and H implies that

$$\alpha A(H/K)^{1-\alpha} - \rho = A(H/K)^{1-\alpha} - \frac{H}{K\gamma}[\alpha A(H/K)^{1-\alpha} - (1-\alpha)A(H/K)^{-\alpha}] - \frac{H}{K}[\alpha A(H/K)^{1-\alpha} - \rho]. \tag{50}$$

Equation (50) implicitly determines the steady-state ratio of H to K. If $\gamma = 0$, the steady-state ratio of H to K equals $\frac{1-\alpha}{\alpha}$. Equation (47) implies that $H/K > \frac{1-\alpha}{\alpha}$ if $\gamma > 0$. Equation (46) indicates that the steady-state growth rate of the economy depends positively on H/K. Thus the presence of human capital in utility increases the steady-state growth rate.[9]

Due to a relatively complicated structure of (50), it is not trivial to see whether the steady-state growth rate is monotonically increasing with γ. I solved (50) numerically for the following baseline specification: $\alpha = 0.3$, $\rho = 0.03$, and $A = 0.18$ (the choice of A is such that the implied growth rates are not unrealistic). The dependence of H/K on γ is presented in Table 4. This table also presents the implied steady-state growth rates, g (δ is assumed equal to 0.05). The dependence

[9] Because the underlying model is symmetric regarding human and physical capital, one can expect that the presence of physical capital in utility would also increase the steady-state growth rate.

Table 4: The dependence of the steady-state H/K ratio and the steady-state growth rate on the human-capital parameter in utility in a one-sector model for the baseline specification of parameters.

γ	0	0.2	0.4	0.6	0.8	1	2	3	∞
H/K	2.3	2.5	2.6	2.7	2.8	2.9	3.0	3.1	3.4
g (%)	1.8	2.3	2.6	2.9	3.1	3.2	3.7	3.9	4.6

of H/K and g on γ is clearly positive. Similarly to the extended AK model, the effect of γ on g is important even for relatively small values of γ. I also examined the dependence of H/K and g on γ for the following one-variable departures from the baseline specification: $A = 0.16$, $A = 0.20$, $\alpha = 0.25$, $\alpha = 0.35$, $\alpha = 0.40$, $\rho = 0.02$, and $\rho = 0.04$. For all of these specifications, H/K and g depended positively on γ for the values of γ shown in Table 4.

3.3 Two sectors with human capital in utility

This subsection presents a two-sector growth model that is derived from the Uzawa-Lucas model. The present model effectively extends the Uzawa-Lucas model by including human capital in utility. The analysis is focused on the steady-state behavior of the model. Utility takes the same form as in the two previous models. Physical capital is produced by a Cobb-Douglas production function, whereas the human-capital sector uses human capital as the only factor of production. Utility is maximized subject to the following dynamic budget constraints:

$$\dot{K} = AK^{\alpha}(uH)^{1-\alpha} - C - \delta K, \qquad (51)$$

$$\dot{H} = B(1-u)H - \delta H, \qquad (52)$$

where u is a variable between 0 and 1, and B is a positive parameter. K and H are state variables, whereas u and C are control variables. The present-value Hamiltonian is

$$\mathcal{H} = e^{-\rho t}(\ln C + \gamma \ln H) + \lambda_K[AK^{\alpha}(uH)^{1-\alpha} - C - \delta K] + \lambda_H[B(1-u)H - \delta H].$$

The first-order conditions are

$$e^{-\rho t}/C = \lambda_K, \tag{53}$$

$$\lambda_K A K^\alpha H^{1-\alpha}(1-\alpha)u^{-\alpha} = \lambda_H B H, \tag{54}$$

$$\dot{\lambda}_K = -\frac{\partial \mathcal{H}}{\partial K} = \lambda_K[-A\alpha K^{\alpha-1}(uH)^{1-\alpha} + \delta], \tag{55}$$

$$\dot{\lambda}_H = -\frac{\partial \mathcal{H}}{\partial H} = -e^{-\rho t}\gamma/H - \lambda_K A K^\alpha u^{1-\alpha}(1-\alpha)H^{-\alpha} - \lambda_H[B(1-u) - \delta]. \tag{56}$$

The transversality conditions are

$$\lim_{t\to\infty} \lambda_K K = 0, \tag{57}$$

$$\lim_{t\to\infty} \lambda_H H = 0. \tag{58}$$

In the steady state, all variables grow at constant rates. From (52) it follows that u is constant in the steady state. Equation (55) then implies that H/K is constant in the steady state. In other words, the growth rate of K equals the growth rate of H, which also equals the growth rate of output. Equation (51) implicates that the C/K ratio is constant in the steady state. Thus the growth rate of consumption equals the growth rate of physical capital, human capital, and output. Let g denote this growth rate. From (53) it follows that

$$g = -\rho - \frac{\dot{\lambda}_K}{\lambda_K}. \tag{59}$$

Equation (54) implies that the growth rates of λ_K and λ_H are identical. If λ_K is expressed from (54) and substituted in (56), we obtain

$$\dot{\lambda}_H = -\frac{e^{-\rho t}\gamma}{H} - \lambda_H(B-\delta). \tag{60}$$

Equations (53), (54), and (60) imply

$$\frac{\dot{\lambda}_H}{\lambda_H} = -\frac{C\gamma B}{AK^\alpha H^{1-\alpha}(1-\alpha)u^{-\alpha}} - (B-\delta). \tag{61}$$

From equations (59) and (61) it follows that

$$g = B - \delta - \rho + \frac{\gamma BC/K}{AK^{\alpha-1}H^{1-\alpha}(1-\alpha)u^{-\alpha}}. \tag{62}$$

Equation (51) implicates that

$$C/K = AK^{\alpha-1}u^{1-\alpha}H^{1-\alpha} - \delta - g. \tag{63}$$

From (55) and (59) we obtain

$$A\alpha K^{\alpha-1}H^{1-\alpha}u^{1-\alpha} = \rho + g + \delta. \tag{64}$$

If (63) and (64) are substituted in (62), we have

$$g = B - \delta - \rho + \frac{\gamma Bu\alpha[(\rho+g+\delta)/\alpha - \delta - g]}{(1-\alpha)(\rho+g+\delta)}. \tag{65}$$

From (52) it follows that

$$u = \frac{B-\delta-g}{B}. \tag{66}$$

If this is substituted in (65), we get

$$g = B - \delta - \rho + \frac{\gamma(B-\delta-g)\alpha[(\rho+g+\delta)/\alpha - \delta - g]}{(1-\alpha)(\rho+g+\delta)}. \tag{67}$$

In the standard Uzawa-Lucas model with no human capital in utility ($\gamma = 0$), the growth rate equals $B - \delta - \rho$. From (67) we see that in the present model the growth rate is higher, although it should not exceed $B - \delta$ (if g exceeded $B - \delta$, the fraction on the right-hand side of (67) would be negative, and, consequently, the right-hand side of (67) would be below $B - \delta$, which is a contradiction). From (67) it follows that g can be expressed in a quadratic equation. If only positive roots are accepted, the solution to this quadratic equation can be written as follows:

$$g = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2a_1}, \tag{68}$$

where

$$a_1 = 1 + \gamma, \tag{69}$$

$$a_2 = 2\rho + 2\delta - B + \delta\gamma - B\gamma + \gamma\frac{\rho+\delta-\delta\alpha}{1-\alpha}, \tag{70}$$

$$a_3 = -(B-\delta-\rho)(\rho+\delta) - \gamma(B-\delta)\frac{\rho+\delta-\delta\alpha}{1-\alpha}. \tag{71}$$

To see how the growth rate depends on γ, I tabulated the growth rates for the following baseline specification: $B = 0.1$, $\delta = 0.05$, $\rho = 0.03$, and $\alpha = 0.3$. Table 5 presents the growth rates for selected values of γ. The growth rate depends

Table 5: The dependence of the steady-state growth rate on the human-capital parameter in utility in the extended Uzawa-Lucas model for the baseline specification of parameters.

γ	0	0.2	0.4	0.6	0.8	1	2	3	∞
g (%)	2.0	2.5	2.9	3.2	3.4	3.6	4.1	4.3	5.0

positively on γ, but it stays below $B - \delta = 0.05$ for finite values of γ. Similarly to the previous models, the effect of γ on g is not negligible even if γ is relatively small.

I also checked whether g depends positively on γ for the following one-variable departures from the baseline specification: $B = 0.09$, $B = 0.11$, $B = 0.12$, $\delta = 0.04$, $\delta = 0.06$, $\rho = 0.02$, $\rho = 0.04$, $\alpha = 0.25$, $\alpha = 0.35$, and $\alpha = 0.40$. For these specifications, the dependence of g on γ turned out to be positive for the values of γ shown in Table 5.

3.4 Two sectors with physical capital in utility

This subsection considers an extension of the Uzawa-Lucas model in which physical capital is included in utility. The analysis is focused on the steady-state dynamics of the model. The present-value Hamiltonian is

$$e^{-\rho t}(\ln C + \mu \ln K) + \lambda_K[AK^\alpha(uH)^{1-\alpha} - C - \delta K] + \lambda_H[B(1-u)H - \delta H],$$

where μ is a positive parameter, λ_K is the marginal shadow value of physical capital, and λ_H is the marginal shadow value of human capital. The first-order conditions are

$$e^{-\rho t}/C = \lambda_K, \tag{72}$$

$$\lambda_K AK^\alpha H^{1-\alpha}(1-\alpha)u^{-\alpha} = \lambda_H BH, \tag{73}$$

$$\dot{\lambda}_K = -\frac{\partial \mathcal{H}}{\partial K} = -\frac{e^{-\rho t}\mu}{K} + \lambda_K[-A\alpha K^{\alpha-1}(uH)^{1-\alpha} + \delta], \tag{74}$$

$$\dot{\lambda}_H = -\frac{\partial \mathcal{H}}{\partial H} = -\lambda_K AK^\alpha u^{1-\alpha}(1-\alpha)H^{-\alpha} - \lambda_H[B(1-u) - \delta]. \tag{75}$$

From the equation of motion for H it follows that u is constant in the steady state. The equations of motion for λ_H and λ_K can be simplified in the following way:

$$\frac{\dot{\lambda}_H}{\lambda_H} = -(B - \delta), \tag{76}$$

$$\frac{\dot{\lambda}_K}{\lambda_K} = -\frac{\mu C}{K} - A\alpha K^{\alpha-1}(uH)^{1-\alpha} + \delta. \tag{77}$$

The growth rate of λ_K is constant if C/K and H/K are constants. It then follows from (73) that λ_K and λ_H grow at the same rate. Hence,

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{H}}{H} = B - \delta - \rho. \tag{78}$$

Thus the presence of physical capital in utility does not affect the long-run growth rate of the economy.

4 Three types of capital

This section considers a model with three types of capital: physical capital, human capital (both used in the production of physical goods), and educational capital (used in the production of human capital and educational capital). The educational capital is given by the quantity and quality of teachers, professors, and scientists and reflects the general access to education in a given economy. It is assumed that the access to education increases the quality of life in a given economy, i.e., the economy-wide level of the educational capital (denoted by S) is included in utility.[10] The planner's problem is

$$\max_{C, I_S} \int_0^\infty e^{-\rho t}(\ln C + \gamma \ln S)dt$$

subject to

$$\dot{K} = AK^\alpha H^{1-\alpha} - C - \delta K, \tag{79}$$

$$\dot{H} = BS - I_S - \delta H, \tag{80}$$

$$\dot{S} = I_S - \delta S, \tag{81}$$

[10] One can also think that S contributes to the level of culture in a given society. In addition, the setup of the problem is consistent with the notion that S reflects access to healthcare. It is plausible that people value the access to healthcare.

where I_S is gross investment in the educational capital.[11] For simplicity, the depreciation rates are assumed the same for all types of capital.

Assume first that the educational capital does not contribute to utility ($\gamma = 0$).[12] The present-value Hamiltonian is now

$$\mathcal{H} = e^{-\rho t} \ln C + \lambda_K (AK^\alpha H^{1-\alpha} - C - \delta K) + \lambda_H (BS - I_S - \delta H) + \lambda_S (I_S - \delta S). \quad (82)$$

The first-order conditions are

$$\frac{e^{-\rho t}}{C} = \lambda_K, \quad (83)$$

$$\lambda_H = \lambda_S, \quad (84)$$

$$\dot{\lambda}_K = \lambda_K (\delta - A\alpha K^{\alpha-1} H^{1-\alpha}), \quad (85)$$

$$\dot{\lambda}_H = -\lambda_K A K^\alpha (1-\alpha) H^{-\alpha} + \delta \lambda_H, \quad (86)$$

$$\dot{\lambda}_S = -\lambda_H B + \lambda_S \delta. \quad (87)$$

The transversality conditions are

$$\lim_{t\to\infty} \lambda_K K = 0, \quad (88)$$

$$\lim_{t\to\infty} \lambda_H H = 0, \quad (89)$$

$$\lim_{t\to\infty} \lambda_S S = 0. \quad (90)$$

From (85) it follows that K/H is constant in the steady state. The first-order conditions imply that

$$\frac{\dot{\lambda}_H}{\lambda_H} = \delta - B = \delta - \frac{\lambda_K}{\lambda_H} A K^\alpha (1-\alpha) H^{-\alpha}. \quad (91)$$

The constancy of the growth rate of λ_H requires a constancy of the ratio of λ_K to λ_H. Thus the growth rate of the economy is given by

$$\frac{\dot{H}}{H} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = B - \delta - \rho. \quad (92)$$

[11] This model resembles the variant of the extended Uzawa-Lucas model in which the part of human capital that is employed in the educational sector enters into utility. The difference is that in the present model people benefit from the education of others (the economy-wide level of S).

[12] This case corresponds to the decentralized solution because individuals cannot affect the economy-wide level of S.

Assume now that $\gamma > 0$. The condition (87) is now modified:

$$\dot{\lambda}_S = -\lambda_H B + \lambda_S \delta - \frac{e^{-\rho t}\gamma}{S} \tag{93}$$

Thus the presence of S in utility decreases the growth rate of λ_S (or, equivalently, increases the growth rate of the economy). Again, K/H and λ_K/λ_H are constants along the steady state. The equality of the growth rates of λ_K and λ_H implies that

$$\lambda_H = \lambda_K \frac{(1-\alpha)K}{\alpha H}. \tag{94}$$

The growth rate of λ_H now can be expressed in two ways:

$$\frac{\dot{\lambda}_H}{\lambda_H} = \delta - B - \frac{C\gamma\alpha H}{(1-\alpha)SK}, \tag{95}$$

$$\frac{\dot{\lambda}_H}{\lambda_H} = \delta - \alpha A K^{\alpha-1} H^{1-\alpha}. \tag{96}$$

Equation (95) implies that the growth rate of S equals the growth rate of C. The equation of motion for physical capital implies that the growth rate of C equals the growth rate of K in the steady state. Let g denote this growth rate of the economy. The equality of the growth rates of H and S implies that

$$I_S = \frac{BS^2}{S+H}. \tag{97}$$

From this it follows that

$$g = \frac{B}{1+H/S} - \delta. \tag{98}$$

Thus the growth rate of the economy does not exceed $B - \delta$. From equations (83), (95), and (96) it follows that

$$g = B - \delta - \rho + \frac{C\gamma\alpha H}{(1-\alpha)SK}, \tag{99}$$

$$g = \alpha A K^{\alpha-1} H^{1-\alpha} - \delta - \rho. \tag{100}$$

Equation (99) implies that g is higher than in a corresponding model with no capital in utility. The equation of motion for physical capital can be rewritten as

$$g = AK^{\alpha-1}H^{1-\alpha} - C/K - \delta. \tag{101}$$

Table 6: The dependence of the steady-state growth rate on γ for the baseline specification in the planner's solution to the three-capital model.

γ	0	0.2	0.4	0.6	0.8	1	2	3	∞
g (%)	2.0	2.7	3.1	3.4	3.6	3.8	4.2	4.4	5.0

From the above equations, g can be expressed in a quadratic equation equivalent to

$$g = B - \delta - \rho + \gamma \frac{\alpha}{1-\alpha}[(g+\delta+\rho)/\alpha - g - \delta][B/(g+\delta) - 1]. \tag{102}$$

The positive solution takes the form

$$g = \frac{-a_2 + \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}, \tag{103}$$

where

$$a_1 = 1 + \gamma, \tag{104}$$

$$a_2 = 2\delta + \rho - B + \delta\gamma - B\gamma + \gamma[(\rho+\delta)/(1-\alpha) - \delta\alpha/(1-\alpha)], \tag{105}$$

$$a_3 = -(B - \delta - \rho)\delta - \gamma(B-\delta)[(\rho+\delta)/(1-\alpha) - \delta\alpha/(1-\alpha)]. \tag{106}$$

The result resembles that of the Uzawa-Lucas model with human capital in utility. Table 6 shows the dependence of g on γ for the baseline specification $B = 0.1$, $\delta = 0.05$, $\rho = 0.03$, and $\alpha = 0.3$. The dependence is clearly positive.

I also considered the following one-variable departures from the baseline specification: $B = 0.09$, $B = 0.11$, $B = 0.12$, $\delta = 0.04$, $\delta = 0.06$, $\rho = 0.02$, $\rho = 0.04$, $\alpha = 0.25$, $\alpha = 0.35$, and $\alpha = 0.40$. For all of these specifications, the dependence of g on γ was positive if γ took the values shown in Table 6.

It is straightforward to show that the socially optimal solution is achieved in the decentralized setup (in which the economy-wide level of S enters into utility but is taken as given) if S is subsidized at an appropriate rate τ. Assuming this subsidy is financed by lump-sum taxes, the problem is then equivalent to

$$\max_{C, I_S} \int_0^\infty e^{-\rho t} \ln C dt$$

subject to

$$\dot{K} = AK^\alpha H^{1-\alpha} - C - \delta K + \tau S - T, \tag{107}$$

$$\dot{H} = BS - I_S - \delta H, \tag{108}$$

$$\dot{S} = I_S - \delta S, \tag{109}$$

where the balanced-budget condition requires that $\tau S = T$ on the economy-wide level. The first-order conditions lead to

$$B + \tau \frac{H\alpha}{K(1-\alpha)} = \alpha A H^{1-\alpha} K^{\alpha-1}, \tag{110}$$

$$g = \alpha A H^{1-\alpha} K^{\alpha-1} - \delta - \rho. \tag{111}$$

From this it follows that the presence of subsidies to S increases the steady-state growth rate of the economy. Subsidies to S can be chosen such that the socially optimal growth rate of the economy is achieved. The ratios of all relevant variables then correspond to the socially optimal case. The socially optimal growth rate can also be achieved if subsidies to S are financed by taxes on output. However, it can be shown that the ratios H/K and C/K do not correspond to the socially optimal ratios in that case.

5 Conclusion

This paper examines growth models in which some types of capital directly contribute to utility. The inclusion of capital in utility seems especially attractive for human capital: it is plausible that health and education improve the quality of life even if one abstracts from their effects on income. One can also consider that in the real world with uncertainty, both human and physical capital increase welfare by providing a certain level of security.

The inclusion of capital in utility has several interesting effects. In the extended AK model, the growth rate is unambiguously increased if (composite) capital is included in utility. On the other hand, in the extended Ramsey model with logarithmic utility, the speed of convergence is decreased if capital enters into utility. In addition, the paper considers extensions of standard one-sector and two-sector growth models with multiple capital goods and additively logarithmic utility. The inclusion of human capital in utility (as well as the inclusion of physical capital in utility) increases the steady-state growth rate in a one-sector two-capital growth model in which both types of capital are productive. In the extended Uzawa-Lucas model, the presence of human capital in utility increases the steady-state growth rate, while the presence of physical capital in utility leaves the steady-state growth rate unaffected. The steady-state growth rate is increased in the planner's solution to a three-capital two-sector model based on the Uzawa-Lucas model if people value access to education (or access to health-care). In this model, the growth rate in the decentralized setup is suboptimal; the socially optimal solution can be achieved by subsidizing the educational capital if the subsidies are financed by lump-sum taxes; however, the socially optimal allocation is not achieved if the subsidies are financed by taxes on output.

Additionally, the paper shows that an extended AK model with human capital in utility can exhibit an empirically plausible imbalance effect between human and physical capital if the initial ratio of human to physical capital differs from its steady-state value. If the model were extended with adjustment costs and irreversibility restrictions for investment, it would predict a positive dependence of output growth on the initial ratio of human to physical capital, which is a more realistic implication than a corresponding U-shaped dependence in the one-sector two-capital model with no human capital in utility. The extended AK model is imperfect in the sense that human capital is not productive. If human capital is productive, the presence of human capital in utility improves the initial imbalance effect if we accept the conjecture that the steady-state solution is the only solution in the one-sector two-capital model with human capital in utility.

Appendix A

Let the instantaneous utility function in the extended AK model be

$$U(C, K) = \frac{(C^{1-\nu}K^{\nu})^{1-\theta} - 1}{1-\theta}, \tag{112}$$

where ν is a parameter between 0 and 1, and $\theta > 0$ is the inverse of the elasticity of intertemporal substitution of the composite of C and K. The present-value Hamiltonian is

$$\mathcal{H} = e^{-\rho t}\frac{(C^{1-\nu}K^{\nu})^{1-\theta} - 1}{1-\theta} + \lambda(AK - \delta K - C). \tag{113}$$

The first-order conditions are

$$e^{-\rho t}K^{\nu(1-\theta)}(1-\nu)C^{(1-\nu)(1-\theta)-1} = \lambda, \tag{114}$$

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial K} = -e^{-\rho t} C^{(1-\nu)(1-\theta)} K^{\nu(1-\theta)-1} \nu + \lambda(\delta - A). \tag{115}$$

From the first-order conditions it follows that

$$\frac{\dot{\lambda}}{\lambda} = -\rho + \nu(1-\theta)\frac{\dot{K}}{K} + [(1-\nu)(1-\theta) - 1]\frac{\dot{C}}{C}, \tag{116}$$

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\nu}{1-\nu}\frac{C}{K} + \delta - A, \tag{117}$$

$$\frac{\dot{C}}{C}[(1-\nu)(1-\theta)-1] = \rho - \nu(1-\theta)(A-\delta) + \delta - A + \nu(1-\theta)\frac{C}{K} - \frac{\nu}{1-\nu}\frac{C}{K}. \tag{118}$$

The equation of motion for capital is

$$\frac{\dot{K}}{K} = A - \delta - \frac{C}{K}. \tag{119}$$

The growth rate of C depends positively on C/K, whereas the growth rate of K depends negatively on C/K. If C/K departs from its steady-state value, it further deviates from it. Thus the steady-state solution is the only solution. The equality of the growth rates of C and K implies that

$$\frac{C}{K} = \frac{\rho - (1-\theta)(A-\delta)}{\nu/(1-\nu) + \theta}. \tag{120}$$

The right-hand side must be positive and it depends negatively on ν. Thus the growth rate of the economy depends positively on ν.

Appendix B

This appendix considers utility given by (112) in the extended Ramsey model. The present-value Hamiltonian is

$$\mathcal{H} = e^{-\rho t}\frac{(C^{1-\nu}K^{\nu})^{1-\theta} - 1}{1-\theta} + \lambda(AK^{\alpha} - \delta K - C). \tag{121}$$

From the first-order conditions it follows that

$$\frac{\dot{C}}{C}[(1-\nu)(1-\theta)-1] = \rho + \delta + \nu(1-\theta)\delta - [\nu(1-\theta)+\alpha]AK^{\alpha-1} + \frac{C}{K}[\nu(1-\theta) - \nu/(1-\nu)]. \tag{122}$$

Consumption and capital are constant in the steady state. The steady-state value of capital, K^*, satisfies

$$AK^{*\alpha-1} = \frac{\rho + \delta + \delta\nu/(1-\nu)}{\alpha + \nu/(1-\nu)}. \tag{123}$$

The log-linearized equations of motion are now

$$\frac{\dot{K}}{K} = (A\alpha K^{*\alpha-1} - \delta)\ln(K/K^*) - (AK^{*\alpha-1} - \delta)\ln(C/C^*), \tag{124}$$

$$\frac{\dot{C}}{C}[(1-\nu)(1-\theta)-1] = \{AK^{*\alpha-1}[-\alpha\nu(1-\theta)+\alpha(1-\alpha)+\nu/(1-\nu)]+$$
$$\delta[\nu(1-\theta)-\nu/(1-\nu)]\}\ln(K/K^*)+[\nu(1-\theta)-\nu/(1-\nu)][AK^{*\alpha-1}-\delta]\ln(C/C^*). \tag{125}$$

The speed of convergence equals the absolute value of the negative eigenvalue of the Jacobian matrix and is given by

$$\beta = \frac{\sqrt{\rho^2 + 4(1-\alpha)\frac{\rho+\delta-\alpha\delta}{\alpha+\nu/(1-\nu)}\frac{\rho+\delta+\delta\nu/(1-\nu)}{1-(1-\nu)(1-\theta)}} - \rho}{2}. \tag{126}$$

The negative dependence of β on ν is still satisfied if θ is not much larger than 1. For example, for the specification $\rho = 0.03$, $\alpha = 0.75$, $\delta = 0.05$, and $\theta = 2$, β decreases with ν initially, and it starts increasing with ν only if $\nu > 0.72$. Nevertheless, the dependence of β on ν is not strong; $\beta = 1.31\%$ is $\nu = 0$ and $\beta \to 1.25\%$ if $\nu \to 1$. If, however, $\theta = 3$, then the given specification of the other parameters leads to a positive association between β and ν if $\nu > 0.10$. Coefficient β then changes from 0.96% to 1.25% as ν changes from 0 to 1.

References

Barro, R.J. (1991). Economic growth in a cross section of countries. *Quarterly Journal of Economics*, 106, 407-43.

Barro, R.J. and Sala-i-Martin, X. (1995). *Economic Growth*, McGraw-Hill, Boston, MA.

Duczynski, P. (2002). Adjustment costs in a two-capital growth model. *Journal of Economic Dynamics and Control*, 26, 837-850.

Duczynski, P. (2003). On the empirics of the imbalance effect. *International Journal of Business and Economics*, 2, 121-128.

Filer, R.K., Jurajda, S., and Plánovský, J. (1999). Education and wages in the Czech and Slovak Republics during transition. *Labour Economics*, 6, 581-593.

Hirshleifer, J. (1987). *Economic Behavior in Adversity*, Chicago, University of Chicago Press.

King, R.J., Plosser, C.I., and Rebelo, S.T. (1988). Production, growth, and business cycles I. The basic neoclassical model. *Journal of Monetary Economics*, 21, 195-232.

Kydland, F.E. and Prescott, E.C. (1982). Time to build and aggregate fluctuations. *Econometrica*, 50, 1345-1370.

Lucas, R.E., Jr. (1988). On the mechanics of economic development. *Journal of Monetary Economics*, 22, 3-42.

Narodnoye chozyaystvo SSSR v 1982 g. (National Economy of the U.S.S.R. in 1982) . (1983). Finansy i statistika, Moscow, U.S.S.R.

Ortigueira, S. and Santos, M.S. (1997). On the speed of convergence in endogenous growth models. *American Economic Review*, 87, 383-399.

Sidrauski, M. (1967). Rational choice and patterns of growth in a monetary economy. *American Economic Review*, 57, 534-544.

United Nations Development Programme. *Human Development Report*, New York, Oxford, Oxford University Press.

Uzawa, H. (1965). Optimum technical change in an aggregative model of economic growth. *International Economic Review*, 6, 18-31.