On Selected Aspects of the Uzawa-Lucas Model

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Abstract

The Uzawa-Lucas model is one of the most important endogenous growth models. We derive the steady-state growth rate for a general production function in the physical capital sector. We also find a system of equations of motion describing the transitional dynamics in this general case, and solve this system in the log-linear approximation for the Cobb-Douglas production function in the physical capital sector.

Keywords: Endogenous growth; Human capital; Physical capital; Steady state; Transitional dynamics

JEL classification: O41

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This work was partially supported by the Grant Agency of the Czech Republic, grant number 402/04/0642.

1 Introduction

Endogenous growth models have been developed quite extensively in the recent economic literature. In these models, the long-run growth is determined endogenously, within the models. The Uzawa-Lucas model (see Uzawa, 1965, and Lucas, 1988) is an important and classic example of these models. It considers two sectors - one sector producing physical capital goods (which can be converted into consumption), and the other sector producing human capital. The production function of the physical capital sector is typically Cobb-Douglas in physical and human capital, while the production of human capital depends only on human capital.

An interesting extension of the Uzawa-Lucas model has been developed by Rebelo (1991). Rebelo assumes that both human and physical capital are used in the production of human capital. He shows that the observed cross-country disparities in output growth rates can be explained by differences in government policy. Mulligan and Sala-i-Martin (1993) examine transitional dynamics in two-sector models. One of their fundamental results is that the Uzawa-Lucas model exhibits an empirically plausible imbalance effect between human and physical capital: the output growth depends positively on the ratio of human capital to physical capital. Duczynski (2005) goes further in this analysis and observes that this result critically depends on the model's parameters. Xie (1994) deals with the Uzawa-Lucas model and shows that this model can exhibit multiple equilibria if there are large externalities from human capital in the physical capital sector. Lucas (1988) has considered externalities from human capital; thus we may have the Uzawa-Lucas model with or without externalities. Benhabib and Perli (1994) and Ladrón-de-Guevara, Ortigueira, and Santos (1997) demonstrate that multiple steady states may also result from the inclusion of labor-leisure choice.

The present paper discusses some variants of the Uzawa-Lucas model without externalities. We derive the steady-state growth rate for a generalized Uzawa-Lucas model in which the production function in the physical capital sector takes a general constant-returns-to-scale form. We present equations of motion describing the transitional dynamics in this case, and solve these equations for the Cobb-Douglas production function in the log-linear approximation.

2 The model

Before developing the model, we will prove the following proposition:

Proposition: Let $Y = F(K, H)$ be a constant-returns-to-scale (CRTS) production function, other than Cobb-Douglas, where Y is output, K is physical capital, and H is human capital. If Y, K, and H grow at constant rates, these rates should be identical.

Proof: A CRTS production function satisfies $F(\lambda K, \lambda H) = \lambda F(K, H)$ for any positive parameter λ. This function also satisfies the Euler formula:

$$Y = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial H} H \tag{1}$$

The marginal products are intensive variables and should be constant in the steady state. From this it follows that

$$g_Y = \frac{\dot{Y}}{Y} = \frac{\frac{\partial F}{\partial K}\dot{K} + \frac{\partial F}{\partial H}\dot{H}}{\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial H}H}, \tag{2}$$

$$g_Y \left(\frac{\partial F}{\partial K} K + \frac{\partial F}{\partial H} H \right) = \frac{\partial F}{\partial K} K g_K + \frac{\partial F}{\partial H} H g_H, \tag{3}$$

where g_K and g_H are the growth rates of physical capital and human capital, respectively. The last equation implies that

$$g_Y = \pi g_K + (1 - \pi) g_H, \tag{4}$$

where π is the share of physical capital. If the growth rates g_Y, g_K, and g_H are to be constant, also the capital shares should be constant during the process of growth. The Cobb-Douglas production function is the only CRTS function for which the capital share is constant, i.e., independent of the amounts of factors of production (see Duczynski, 2001). Consequently, one possibility is that the production function is Cobb-Douglas. In this proposition we have assumed that the production function is not Cobb-Douglas. We will prove the following lemma:

Lemma: In a CRTS production function $F(K, H)$, the shares of physical and human capital are unique functions of the ratio H/K.

Proof: The share of physical capital is defined as $(\frac{\partial F}{\partial K} K)/Y$. If we multiply both K and H by an arbitrary positive parameter λ, this share is unchanged, Q.E.D.

Therefore, we have proven that the ratio H/K should be fixed during the process of growth. Thus the growth rate of H equals the growth rate of K, and from (4) these rates should equal g_Y, Q.E.D.

Now we come to the model. Assume there are no externalities and no population growth. The problem is

$$\max_{C,u} \int_0^\infty \ln C e^{-\rho t} dt$$

subject to

$$\dot{K} = F(K, uH) - C - \delta K, \tag{5}$$

$$\dot{H} = B(1-u)H - \delta H, \tag{6}$$

where C is consumption, K is physical capital, H is human capital (extensive variables), u is the fraction of human capital employed in the physical capital sector (an intensive variable), B is a fixed technological parameter, ρ is the rate of time preference, and δ is the depreciation rate of physical and human capital. The present-value Hamiltonian is

$$\mathcal{J} = \ln C e^{-\rho t} + \lambda_K [F(K, uH) - C - \delta K] + \lambda_H [B(1-u)H - \delta H], \tag{7}$$

where λ_K and λ_H are co-state variables. The first-order conditions are

$$e^{-\rho t}/C = \lambda_K, \tag{8}$$

$$\lambda_K F_2 = \lambda_H B, \tag{9}$$

$$\dot{\lambda}_K = -\lambda_K (F_1 - \delta), \tag{10}$$

$$\dot{\lambda}_H = \lambda_H [\delta - B(1-u)] - \lambda_K F_2 u = \lambda_H (\delta - B), \tag{11}$$

where F_1 and F_2 are the partial derivatives of F with respect to the inputs of production (the marginal products). Intensive variables should be constant in the steady state. The marginal products are intensive variables. From (9) it follows that the growth rate of λ_K should be equal to the growth rate of λ_H in the steady state. Equation (8) implies that

$$\frac{\dot{C}}{C} = -\frac{\dot{\lambda}_K}{\lambda_K} - \rho. \tag{12}$$

Thus, the steady-state growth rate of consumption equals $B - \delta - \rho$. The equation of motion for physical capital implies that the steady-state growth rate of consumption equals the steady-state growth rate of physical capital since the ratio of these two variables should be fixed in the steady state. In a steady-state, all extensive variables should grow at constant rates. From the proposition presented at the beginning of this section it follows that the steady state growth rate of physical capital should be equal to the steady-state growth rate of human capital, which should be equal to the steady-state output growth rate. Therefore,

$$g^* = B - \delta - \rho, \tag{13}$$

where g^* is the steady-state growth rate of the economy. We have shown that this growth rate does not depend on the production function in the physical capital sector.

The transversality conditions are

$$\lim_{t \to \infty} \lambda_K K = 0, \tag{14}$$

$$\lim_{t \to \infty} \lambda_H H = 0. \tag{15}$$

It is convenient to introduce intensive variables $x = H/K$ and $y = C/K$. The production function can be re-written in the following way:

$$F(K, uH) = KF(1, uH/K) = Kf(ux), \tag{16}$$

where $f(.) = F(1, .)$. The equation of motion for physical capital is

$$\frac{\dot{K}}{K} = f(ux) - y - \delta. \tag{17}$$

The marginal products satisfy:

$$F_1 = f(ux) - f'(ux)ux, \tag{18}$$

$$F_2 = f'(ux). \tag{19}$$

The first-order conditions imply the following equations of motion for x, y, and u:

$$\frac{\dot{x}}{x} = B - Bu - f(ux) + y, \tag{20}$$

$$\frac{\dot{y}}{y} = -f'(ux)ux - \rho + y, \tag{21}$$

$$\frac{\dot{u}}{u} = [-B + f(ux) - f'(ux)ux]\frac{f'(ux)}{f''(ux)ux} - B + Bu + f(ux) - y. \tag{22}$$

These equations are relatively complicated. We can present relevant formulae corresponding to the CES (constant elasticity of substitution) production function (see Barro and Sala-i-Martin, 1995, p. 43):

$$f(ux) = A\{ab^\psi + (1 - a)[(1 - b)ux]^\psi\}^{1/\psi}, \tag{23}$$

$$f'(ux) = A\{ab^{\psi} + (1-a)[(1-b)ux]^{\psi}\}^{1/\psi-1}(1-a)(1-b)^{\psi}(ux)^{\psi-1}, \tag{24}$$

$$f''(ux) = A(1-a)(1-b)^{\psi}(1/\psi-1)\{ab^{\psi}(ux)^{-\psi}+(1-a)(1-b)^{\psi}\}^{1/\psi-2}ab^{\psi}(-\psi)(ux)^{-\psi-1}, \tag{25}$$

where a and b are parameters between 0 and 1, and $\psi < 1$. The elasticity of substitution equals $1/(1-\psi)$. It is difficult to solve the equations of motion for the CES production function; therefore, we leave this task for future research. It is easier to find a solution for the Cobb-Douglas function:

$$f(ux) = A(ux)^{1-\alpha}, \tag{26}$$

where α is the share of physical capital. The equations of motion for intensive variables are:

$$\frac{\dot{x}}{x} = B - Bu + y - Au^{1-\alpha}x^{1-\alpha}, \tag{27}$$

$$\frac{\dot{y}}{y} = y - \rho + (\alpha - 1)Au^{1-\alpha}x^{1-\alpha}, \tag{28}$$

$$\frac{\dot{u}}{u} = -y + B/\alpha - B + Bu. \tag{29}$$

Intensive variables are constant in the steady state:

$$x^* = \left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}}\frac{B}{\rho}, \tag{30}$$

$$y^* = \rho + \frac{1-\alpha}{\alpha}B, \tag{31}$$

$$u^* = \frac{\rho}{B}. \tag{32}$$

Extensive variables grow at the rate of $g^* = B - \rho - \delta$ in the steady state. The transversality conditions require $\rho > 0$. The equations of motion for intensive variables can be log-linearized around the steady state:

$$\begin{pmatrix} d\ln(x/x^*)/dt \\ d\ln(y/y^*)/dt \\ d\ln(u/u^*)/dt \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & -a_2 \\ a_3 & a_2 & a_3 \\ 0 & -a_2 & a_4 \end{pmatrix} \begin{pmatrix} \ln(x/x^*) \\ \ln(y/y^*) \\ \ln(u/u^*) \end{pmatrix}, \tag{33}$$

where $a_1 = -\frac{1-\alpha}{\alpha}B$, $a_2 = \rho + \frac{1-\alpha}{\alpha}B = y^*$, $a_3 = -\frac{(1-\alpha)^2 B}{\alpha}$, and $a_4 = \rho$. Parameters a_1, a_2, and a_4 are the eigenvalues of the Jacobian matrix. The speed of convergence equals $|a_1| = \frac{1-\alpha}{\alpha}B$. This speed corresponds to the result of Ortigueira and Santos (1997), who operated in a more general framework. The solution to the log-linearized Uzawa-Lucas model is given by the initial condition

$[x(t=0)=x_0]$ and by the components of the eigenvector corresponding to the negative eigenvalue a_1:

$$\ln(x/x^*) = \ln(x_0/x^*)e^{-\frac{1-\alpha}{\alpha}Bt}, \tag{34}$$

$$\ln(y/y^*) = \ln(u/u^*) = \frac{a_3}{a_1 - a_2 - a_3}\ln(x/x^*). \tag{35}$$

Since $\frac{a_3}{a_1-a_2-a_3} > 0$, a high ratio of human to physical capital (x) is connected with high u and high $y = C/K$. This observation corresponds to one of the results in Mulligan and Sala-i-Martin (1993). One could easily derive that high x is associated with low C/H.

3 Conclusion

The Uzawa-Lucas model is one of the most important endogenous growth models. The present paper first examines this model with a general production function in the physical capital sector. It is shown that the long-run growth rate does not depend on the specification of this production function. The paper then derives the equations of motion describing the transitional dynamics in this general case. A possible solution of these equations is complicated; future research can find a solution for the CES production function. At the end of the paper we find a closed-form solution for the log-linearized Uzawa-Lucas model if the production function is Cobb-Douglas, which is a special case of the CES function.

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