# On Steady-State Solutions of Selected Endogenous Growth Models

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**Abstract**

This paper considers three natural extensions of the Uzawa-Lucas model. These extensions work with physical capital and one or two types of human capital. There are two or three sectors in which production takes place. We show that the steady-state growth rate of the economy can be analytically derived for all of these models.

**Keywords:** Endogenous growth; Human capital; Physical capital; Steady state

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# 1 Introduction

Endogenous growth models constitute an important class of models in which the growth rate of the economy does not tend to diminish in the long run. Rebelo (1991) examines the AK growth model, which is the simplest endogenous growth model. In this model production depends linearly on the capital stock; the capital stock is frequently assumed to be a composite of human and physical capital. There are no transitional dynamics in the AK model: the economy is always in its steady state with a fixed growth rate. McGrattan (1998) provides empirical evidence favoring AK-type growth models since there is a strong positive relationship between investment rates and growth rates. Duczynski (2004b) solves the AK model for Stone-Geary, exponential, and quadratic preferences, and also for preferences in which utility depends on consumption, capital, and investment. Romer (1986) and Barro (1990) present certain extensions of the AK model. Romer (1986) introduces a framework with externalities from physical capital and increasing returns to scale. Barro (1990) considers a model with government spending entering the production function.

The Uzawa-Lucas model (see Uzawa, 1965, and Lucas, 1988) is a two-capital two-sector generalization of the AK model. This model works with one sector producing physical capital goods (which can be converted into consumption), and the other sector producing human capital. The production function of the physical capital sector is Cobb-Douglas in physical and human capital, while the production of human capital depends only on human capital. The Uzawa-Lucas model (without externalities) has a globally stable steady state in which consumption, output, human capital, and physical capital grow at the same rate. There are transitional dynamics in this model: if the initial ratio of human capital to physical capital differs from its steady-state value, this ratio gradually converges to the steady state. Ortigueira and Santos (1997) studied the speed of convergence in related endogenous growth models.

Rebelo (1991) extends the Uzawa-Lucas model by assuming that physical capital is used in the production of human capital. He shows that the observed cross-country disparities in output growth rates can be explained by differences in government policy. Caballé and Santos (1993) consider a two-sector model with general linearly homogeneous production functions and demonstrate a global-convergence property. Mulligan and Sala-i-Martin (1993) examine convergence towards a steady state in two-sector models. One of their important results is that the Uzawa-Lucas model exhibits an empirically plausible imbalance effect between human and physical capital: the output growth depends positively on the ratio of human capital to physical capital. Duczynski (2004a) observes that the imbalance effect in the Uzawa-Lucas model is empirically implausible if the physical capital share in the physical capital sector is low and/or if the elasticity of the intertemporal substitution of consumption is low. Xie (1994) shows that the Uzawa-Lucas model can exhibit multiple equilibria if there are large externalities from human capital in the physical capital sector. Benhabib and Perli (1994) and Ladrón-de-Guevara, Ortigueira, and Santos (1997) demonstrate that multiple steady states may also result from the inclusion of labor-leisure choice.

The present paper solves three models which are based on the Uzawa-Lucas model. The first model works with three sectors using one type of physical capital and two types of human capital as inputs. A crucial assumption is that the production of one type of human capital (tertiary education) depends only on this type of capital (university professors must have tertiary education), while the production of the other type of human capital (primary and secondary education) depends on both types of capital (in at least some economies primary school teachers need not have tertiary education). It is shown that the long-run growth rate of the economy depends only on the productivity of the tertiary sector (and depreciation and preference parameters); it does not depend on the productivity of the physical capital sector or the primary and secondary education sector. The second model is basically the Rebelo model with two types of capital (physical and human capital) and two sectors. This model generalizes the Uzawa-Lucas model since the production of human capital depends not only on human capital, but also on physical capital. We derive the result which is briefly presented in Rebelo (1991), and we also extend this model by assuming generally different depreciation rates of human and physical capital. The third model is a natural extension of the Rebelo model in which there are three sectors and three types of capital (like in the first model). This model is relatively complicated; nevertheless, we are still able to find a recipe for a closed-form solution for the steady-state growth rate of the economy if there are identical depreciation rates for all types of capital. It is of some interest to have steady-state solutions in all the three models considered since the given models are quite standard - they are natural extensions of the influential Uzawa-Lucas model.

## 2 Model 1

There is no population growth and no technological change. The production function for physical capital is given by

$$Y = AK^{\alpha}(u_1 H_1)^{\beta}(u_2 H_2)^{1-\alpha-\beta}, \tag{1}$$

where $Y$ is output, $A$ is a fixed technological parameter, $K$ is physical capital, $H_1$ is one type of human capital (the primary and secondary education), $H_2$ is the other type of human capital (the tertiary education), $u_1$ is the fraction of $H_1$ employed in the production of $Y$, $u_2$ is the fraction of $H_2$ employed in the production of $Y$, and $\alpha$, $\beta$, and $1-\alpha-\beta$ are capital shares (fixed parameters between 0 and 1). We assume that both $H_1$ and $H_2$ enter into the production of $H_1$ (in some economies primary school teachers need not have tertiary education), while the production of $H_2$ depends only on $H_2$ (university professors must have tertiary education). The problem is

$$\max_{C,u_1,u_2,v} \int_0^{\infty} \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{K} = AK^{\alpha}(u_1 H_1)^{\beta}(u_2 H_2)^{1-\alpha-\beta} - C - \delta_K K, \tag{2}$$

$$\dot{H}_1 = B_1[(1-u_1)H_1]^{\gamma}(vH_2)^{1-\gamma} - \delta_{H_1} H_1, \tag{3}$$

$$\dot{H}_2 = B_2(1-u_2-v)H_2 - \delta_{H_2} H_2, \tag{4}$$

where $C$ is consumption, $\theta > 0$ is the inverse elasticity of intertemporal substitution, $\rho$ is the rate of time preference, $B_1$ and $B_2$ are fixed technological parameters in the human capital sectors, $v$ is the fraction of $H_2$ employed in the production of $H_1$, $\gamma$ is the capital share of $H_1$ in the production of $H_1$, and $\delta_K$, $\delta_{H_1}$ and $\delta_{H_2}$ are depreciation rates of capital. The present-value Hamiltonian for this problem is

$$\mathcal{J} = \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} + \lambda_K[AK^{\alpha}(u_1 H_1)^{\beta}(u_2 H_2)^{1-\alpha-\beta} - C - \delta_K K] +$$
$$\lambda_{H_1}\{B_1[(1-u_1)H_1]^{\gamma}(vH_2)^{1-\gamma} - \delta_{H_1} H_1\} + \lambda_{H_2}[B_2(1-u_2-v)H_2 - \delta_{H_2} H_2]. \tag{5}$$

There are three state variables ($K$, $H_1$, and $H_2$), four control variables ($C$, $u_1$, $u_2$, and $v$), and three co-state variables ($\lambda_K$, $\lambda_{H_1}$, and $\lambda_{H_2}$). The first-order conditions state that the derivatives of the Hamiltonian with respect to the control variables are zero, and the derivatives of the Hamiltonian with respect to the state variables equal negative time derivatives of co-state variables. The first-order conditions are

$$C^{-\theta}e^{-\rho t} = \lambda_K, \tag{6}$$

$$\lambda_K A K^{\alpha}\beta u_1^{\beta-1}H_1^{\beta}(u_2H_2)^{1-\alpha-\beta} = \lambda_{H_1}B_1\gamma(1-u_1)^{\gamma-1}H_1^{\gamma}(vH_2)^{1-\gamma}, \tag{7}$$

$$\lambda_K A K^{\alpha}(u_1H_1)^{\beta}(1-\alpha-\beta)u_2^{-\alpha-\beta}H_2^{1-\alpha-\beta} = \lambda_{H_2}B_2H_2, \tag{8}$$

$$\lambda_{H_1}B_1[(1-u_1)H_1]^{\gamma}(1-\gamma)v^{-\gamma}H_2^{1-\gamma} = \lambda_{H_2}B_2H_2, \tag{9}$$

$$\dot{\lambda}_K = \lambda_K[\delta_K - A\alpha K^{\alpha-1}(u_1H_1)^{\beta}(u_2H_2)^{1-\alpha-\beta}], \tag{10}$$

$$\dot{\lambda}_{H_1} = -\lambda_K A K^{\alpha}u_1^{\beta}\beta H_1^{\beta-1}(u_2H_2)^{1-\alpha-\beta} + \lambda_{H_1}[\delta_{H_1} - B_1(1-u_1)^{\gamma}\gamma H_1^{\gamma-1}(vH_2)^{1-\gamma}], \tag{11}$$

$$\dot{\lambda}_{H_2} = -\lambda_K A K^{\alpha}(u_1H_1)^{\beta}u_2^{1-\alpha-\beta}(1-\alpha-\beta)H_2^{-\alpha-\beta} -$$

$$\lambda_{H_1}B_1[(1-u_1)H_1]^{\gamma}v^{1-\gamma}(1-\gamma)H_2^{-\gamma} + \lambda_{H_2}[\delta_{H_2} - B_2(1-u_2-v)]. \tag{12}$$

The first-order conditions imply that

$$-\theta\frac{\dot{C}}{C} - \rho = \frac{\dot{\lambda}_K}{\lambda_K}, \tag{13}$$

$$\frac{1-\alpha-\beta}{\beta}\frac{u_1}{u_2} = \frac{1-\gamma}{\gamma}\frac{1-u_1}{v}, \tag{14}$$

$$\dot{\lambda}_{H_1} = \lambda_{H_1}[\delta_{H_1} - B_1(1-u_1)^{\gamma-1}\gamma H_1^{\gamma-1}(vH_2)^{1-\gamma}], \tag{15}$$

$$\dot{\lambda}_{H_2} = \lambda_{H_2}(\delta_{H_2} - B_2). \tag{16}$$

We abstract from corner solutions. Intensive variables ($u_1$, $u_2$, and $v$) must be constant in the steady state (otherwise we would hit the corner solution). Extensive variables ($Y$, $C$, $K$, $H_1$, and $H_2$) should grow at the constant and identical rate in the steady state. The equality of these rates follows from the equations of motion for $K$ and $H_1$. The first-order conditions imply that $\lambda_K$, $\lambda_{H_1}$, and $\lambda_{H_2}$ grow at the same steady-state rate of $\delta_{H_2} - B_2$. The steady-state equation of motion for consumption should equal the steady-state growth rate of the economy. This equation follows from (13). Thus the steady-state growth rate of the economy is given by

$$g^* = \frac{B_2 - \delta_{H_2} - \rho}{\theta}. \tag{17}$$

An important fact following from this result is that the long-run growth rate of the economy does not depend on technological parameters $A$ and $B_1$. We can see the importance of the productivity of the tertiary education sector (parameter $B_2$) for the long-run growth rate of the economy. The transversality condition is

$$\lim_{t\to\infty} \lambda_K K = 0, \tag{18}$$

and similarly for the other types of capital. The transversality condition is equivalent to

$$\frac{B_2 - \delta_{H_2} - \rho}{\theta} < B_2 - \delta_{H_2}. \tag{19}$$

# 3 Model 2 - the Rebelo model

There are two types of capital and two sectors. The Uzawa-Lucas model assumes that only human capital is used in the production of human capital. The present model extends the Uzawa-Lucas model by assuming that both physical capital and human capital are used in the human capital sector. The production of physical capital is given by

$$Y = A(Kv)^{\alpha}(uH)^{1-\alpha}, \tag{20}$$

where $Y$ is the output of physical capital goods, $v$ is the fraction of physical capital employed in the physical capital sector, and $u$ is the fraction of human capital employed in the physical capital sector. The problem is

$$\max_{C,u,v} \int_0^{\infty} \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{K} = A(Kv)^{\alpha}(uH)^{1-\alpha} - C - \delta_K K, \tag{21}$$

$$\dot{H} = B[(1-v)K]^{\beta}[(1-u)H]^{1-\beta} - \delta_H H. \tag{22}$$

A natural assumption is that the education sector is relatively human capital intensive, which means that $\beta < \alpha$. The present-value Hamiltonian equals

$$\mathcal{K} = \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} + \lambda_K[A(Kv)^{\alpha}(uH)^{1-\alpha} - C - \delta_K K] +$$
$$\lambda_H\{B[(1-v)K]^{\beta}[(1-u)H]^{1-\beta} - \delta_H H\}. \tag{23}$$

We again abstract from corner solutions. The first-order conditions are

$$C^{-\theta} e^{-\rho t} = \lambda_K, \tag{24}$$

$$\lambda_K A(Kv)^{\alpha}(1-\alpha)u^{-\alpha}H^{1-\alpha} = \lambda_H B[(1-v)K]^{\beta}H^{1-\beta}(1-\beta)(1-u)^{-\beta}, \quad (25)$$

$$\lambda_K A K^{\alpha}\alpha v^{\alpha-1}(uH)^{1-\alpha} = \lambda_H B K^{\beta}(1-v)^{\beta-1}\beta[(1-u)H]^{1-\beta}, \quad (26)$$

$$\dot{\lambda}_K = -\lambda_K[A\alpha K^{\alpha-1}v^{\alpha}(uH)^{1-\alpha} - \delta_K] - \lambda_H B(1-v)^{\beta}\beta K^{\beta-1}(1-u)^{1-\beta}H^{1-\beta}, \quad (27)$$

$$\dot{\lambda}_H = -\lambda_K A(Kv)^{\alpha}u^{1-\alpha}(1-\alpha)H^{-\alpha} - \lambda_H\{B[(1-v)K]^{\beta}(1-u)^{1-\beta}(1-\beta)H^{-\beta} - \delta_H\}. \quad (28)$$

The transversality conditions are

$$\lim_{t\to\infty} \lambda_K K = 0, \quad (29)$$

$$\lim_{t\to\infty} \lambda_H H = 0. \quad (30)$$

These conditions are equivalent to

$$g^*(1-\theta) < \rho, \quad (31)$$

where $g^*$ is the steady-state growth rate of the economy. The first-order conditions imply that

$$\frac{\alpha}{1-\alpha}\frac{u}{v} = \frac{\beta}{1-\beta}\frac{1-u}{1-v}, \quad (32)$$

$$\frac{\dot{\lambda}_K}{\lambda_K} = \delta_K - AK^{\alpha-1}\alpha v^{\alpha-1}(uH)^{1-\alpha}, \quad (33)$$

$$\frac{\dot{\lambda}_H}{\lambda_H} = \delta_H - B(1-u)^{-\beta}(1-\beta)[(1-v)K]^{\beta}H^{-\beta}. \quad (34)$$

Equation (32) is basically a rational association between $u$ and $v$, and it can be rearranged in the following ways:

$$u = \frac{1}{1 + \frac{\alpha/(1-\alpha)}{\beta/(1-\beta)}(1/v - 1)}, \quad (35)$$

$$v = \frac{1}{1 + \frac{\beta/(1-\beta)}{\alpha/(1-\alpha)}(1/u - 1)}. \quad (36)$$

The association between $u$ and $v$ is increasing. If $u$ tends to 0, also $v$ tends to 0. If $u$ tends to 1, also $v$ tends to 1. If $u$ equals $1-\alpha$, then $v = 1-\beta$. If $u = \beta$, then $v = \alpha$.

Intensive variables ($u$ and $v$) must be constant in the steady state. The equations of motion for $\lambda_K$ and $\lambda_H$ indicate that the steady-state growth rates are identical for $K$ and $H$, and, therefore, also for $Y$. The equation of motion for $K$ implies that also consumption grows at the rate of the other extensive variables. From the first-order conditions it follows that the steady-state growth of $\lambda_K$ equals the steady-state growth of $\lambda_H$. The growth rate of consumption satisfies

$$\frac{\dot{C}}{C} = \frac{-\dot{\lambda}_K/\lambda_K - \rho}{\theta}. \tag{37}$$

The equation of motion for $\lambda_H$ leads to the following expression for the steady-state growth rate of the economy:

$$g^* = \frac{(1-\beta)\Omega - \delta_H - \rho}{\theta}, \tag{38}$$

where $\Omega$ is given by

$$\Omega = B[(1-v)K]^{\beta}[(1-u)H]^{-\beta}. \tag{39}$$

The equation of motion for $H$ leads to

$$g^* = (1-u)\Omega - \delta_H. \tag{40}$$

The equation of motion for $\lambda_K$ implies that

$$g^* = \frac{AB^{(1-\alpha)/\beta}\Omega^{(\alpha-1)/\beta}\alpha(\frac{\beta}{1-\beta}\frac{1-\alpha}{\alpha})^{1-\alpha} - \delta_K - \rho}{\theta}. \tag{41}$$

It is difficult to find the steady-state growth rate of the economy if there are generally different depreciation rates for $K$ and $H$. We can simplify the analysis by assuming that

$$\delta_K = \delta_H = \delta. \tag{42}$$

Equations (38) and (41) then lead to the final formula for the steady-state growth rate of the economy:

$$g^* = \frac{A^{\frac{\beta}{1-\alpha+\beta}}B^{\frac{1-\alpha}{1-\alpha+\beta}}(1-\beta)(\frac{\alpha}{1-\beta})^{\frac{\beta}{1-\alpha+\beta}}(\frac{\beta}{1-\beta}\frac{1-\alpha}{\alpha})^{\frac{(1-\alpha)\beta}{1-\alpha+\beta}} - \delta - \rho}{\theta}. \tag{43}$$

Thus the growth rate of the economy is a positive function of $A$ and $B$, and a negative function of $\delta$, $\rho$, and $\theta$. Rebelo (1991) presents a similar result, although he is not specific regarding the dependence of $g^*$ on $\alpha$ and $\beta$. This dependence is relatively complicated. We can write a simpler formula for $\alpha = \beta$:

$$g^* = \frac{A^{\alpha}B^{1-\alpha}\alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta - \rho}{\theta}, \tag{44}$$

or for $\alpha = 1-\beta$:

$$g^* = \frac{A^{0.5}B^{0.5}\alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta - \rho}{\theta}. \tag{45}$$

An interesting question is how $F(\alpha) = \alpha^\alpha (1-\alpha)^{1-\alpha}$ depends on $\alpha$ if $0 < \alpha < 1$. This function is symmetric around $\alpha = 0.5$. To find a minimum and a maximum of this function, we take a logarithm of $F(\alpha)$, which should have the same maxima and minima as $F(\alpha)$:

$$\ln F(\alpha) = \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha) \tag{46}$$

The derivative of this function with respect to $\alpha$ is $\ln \frac{\alpha}{1-\alpha}$. This derivative is positive if $\alpha > 0.5$ and negative if $\alpha < 0.5$. Therefore, $F(\alpha)$ is decreasing if $\alpha < 0.5$ and increasing if $\alpha > 0.5$. $F(\alpha)$ is not well defined if $\alpha = 0$ or $\alpha = 1$, but its limit value is 1 since

$$\lim_{\alpha \to 0} \alpha^\alpha = 1. \tag{47}$$

The minimum of $F(\alpha)$ is achieved at $\alpha = 0.5$, and the value of this minimum is 0.5.

We can now briefly consider a more general case with arbitrary $\alpha$ and $\beta$ and with $\delta_K$ generally different from $\delta_H$. We are interested in the effect of $\delta_K$ and $\delta_H$ on the growth rate of the economy $g^*$. We cannot express a full analytical formula for $g^*$ in the general case, but we can derive some qualitative results from equations (38) and (41). If we increase $\delta_H$ and fix $\delta_K$, $\Omega$ cannot fall [falling $\Omega$ would mean a decline in $g^*$ from equation (38), but an increase in $g^*$ from equation (41), which is a contradiction]. Therefore, $\Omega$ must increase and $g^*$ decrease [from equation (41)]. This is the same result as in the normal Uzawa-Lucas model, in which the long-run growth rate of the economy depends negatively on the depreciation rate of human capital. If we increase $\delta_K$ and fix $\delta_H$, $\Omega$ cannot increase [an increase in $\Omega$ would mean an increase in $g^*$ from equation (38), but a decrease in $g^*$ from equation (41), which is a contradiction]. Thus, $\Omega$ must go down, and $g^*$ decreases from equation (38). To some extent, this is an expected outcome, but it differs from the standard Uzawa-Lucas model, in which the steady-state growth rate of the economy does not depend on the depreciation rate of physical capital. We can formulate these findings in the following Proposition:

**Proposition:** In the generalized Uzawa-Lucas model (the Rebelo model), the steady-state growth rate of the economy depends negatively on the depreciation rate of human capital, and also negatively on the depreciation rate of physical capital.

We can illustrate this result on a special specification $\alpha = 1 - \beta$. In this case we can analytically derive $\Omega$ and $g^*$ from a quadratic equation resulting from

(38) and (41). $\Omega$ must be positive, so only one root of this equation applies. The result is

$$\Omega = \frac{\delta_H - \delta_K + \sqrt{(\delta_K - \delta_H)^2 + 4(1-\beta)z}}{2(1-\beta)}, \tag{48}$$

$$g^* = \Big(\frac{-\delta_H - \delta_K + \sqrt{(\delta_K - \delta_H)^2 + 4(1-\beta)z}}{2} - \rho\Big)/\theta, \tag{49}$$

where

$$z = AB\alpha\Big[\frac{(1-\alpha)^2}{\alpha^2}\Big]^{1-\alpha}. \tag{50}$$

$g^*$ increases in $z$, and, therefore, in $A$ and $B$. It decreases in $\rho$, and also in $\theta$ if $g^* > 0$. At the same time, $g^*$ really decreases in $\delta_K$ and $\delta_H$, which is consistent with the Proposition presented above.

# 4 Model 3 - a three sector extension of Model 2

This section considers an endogenous growth model with three types of capital: one type of physical capital and two types of human capital. In some sense, this model is a standard model - it is a natural extension of the Rebelo model (or the Uzawa-Lucas model). There are three state and seven control variables. The problem is

$$\max \int_0^\infty \frac{C^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{K} = A(u_1K)^{\alpha_1}(v_1H_1)^{\alpha_2}(w_1H_2)^{1-\alpha_1-\alpha_2} - C - \delta_K K, \tag{51}$$

$$\dot{H}_1 = B_1(u_2K)^{\beta_1}(v_2H_1)^{\beta_2}(w_2H_2)^{1-\beta_1-\beta_2} - \delta_{H_1}H_1, \tag{52}$$

$$\dot{H}_2 = B_2[(1-u_1-u_2)K]^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2} - \delta_{H_2}H_2. \tag{53}$$

The present-value Hamiltonian for this problem is

$$\mathcal{L} = \frac{C^{1-\theta} - 1}{1-\theta} e^{-\rho t} + \lambda_K[A(u_1K)^{\alpha_1}(v_1H_1)^{\alpha_2}(w_1H_2)^{1-\alpha_1-\alpha_2} - C - \delta_K K] +$$

$$\lambda_{H_1}[B_1(u_2K)^{\beta_1}(v_2H_1)^{\beta_2}(w_2H_2)^{1-\beta_1-\beta_2} - \delta_{H_1}H_1] +$$

$$\lambda_{H_2}\{B_2[(1-u_1-u_2)K]^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2} - \delta_{H_2}H_2\}. \tag{54}$$

The first-order conditions are

$$C^{-\theta}e^{-\rho t} = \lambda_K, \tag{55}$$

$$\lambda_K A\alpha_1 u_1^{\alpha_1-1}K^{\alpha_1}(v_1H_1)^{\alpha_2}(w_1H_2)^{1-\alpha_1-\alpha_2} =$$
$$\lambda_{H_2}B_2(1-u_1-u_2)^{\gamma_1-1}\gamma_1 K^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2}, \tag{56}$$

$$\lambda_{H_1}B_1 u_2^{\beta_1-1}\beta_1 K^{\beta_1}(v_2H_1)^{\beta_2}(w_2H_2)^{1-\beta_1-\beta_2} =$$
$$\lambda_{H_2}B_2(1-u_1-u_2)^{\gamma_1-1}\gamma_1 K^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2}, \tag{57}$$

$$\lambda_K A(u_1K)^{\alpha_1}\alpha_2 v_1^{\alpha_2-1}H_1^{\alpha_2}(w_1H_2)^{1-\alpha_1-\alpha_2} =$$
$$\lambda_{H_2}B_2[(1-u_1-u_2)K]^{\gamma_1}\gamma_2(1-v_1-v_2)^{\gamma_2-1}H_1^{\gamma_2}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2}, \tag{58}$$

$$\lambda_{H_1}B_1(u_2K)^{\beta_1}v_2^{\beta_2-1}H_1^{\beta_2}\beta_2(w_2H_2)^{1-\beta_1-\beta_2} =$$
$$\lambda_{H_2}B_2[(1-u_1-u_2)K]^{\gamma_1}\gamma_2(1-v_1-v_2)^{\gamma_2-1}H_1^{\gamma_2}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2}, \tag{59}$$

$$\lambda_K A(u_1K)^{\alpha_1}(v_1H_1)^{\alpha_2}(1-\alpha_1-\alpha_2)w_1^{-\alpha_1-\alpha_2}H_2^{1-\alpha_1-\alpha_2} =$$
$$\lambda_{H_2}B_2[(1-u_1-u_2)K]^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}(1-\gamma_1-\gamma_2)(1-w_1-w_2)^{-\gamma_1-\gamma_2}H_2^{1-\gamma_1-\gamma_2}, \tag{60}$$

$$\lambda_{H_1}B_1(u_2K)^{\beta_1}(v_2H_1)^{\beta_2}(1-\beta_1-\beta_2)w_2^{-\beta_1-\beta_2}H_2^{1-\beta_1-\beta_2} =$$
$$\lambda_{H_2}B_2[(1-u_1-u_2)K]^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}(1-\gamma_1-\gamma_2)(1-w_1-w_2)^{-\gamma_1-\gamma_2}H_2^{1-\gamma_1-\gamma_2}, \tag{61}$$

$$\dot{\lambda}_K = \lambda_K[\delta_K - Au_1^{\alpha_1}K^{\alpha_1-1}\alpha_1(v_1H_1)^{\alpha_2}(w_1H_2)^{1-\alpha_1-\alpha_2}] -$$
$$\lambda_{H_1}B_1u_2^{\beta_1}\beta_1K^{\beta_1-1}(v_2H_1)^{\beta_2}(w_2H_2)^{1-\beta_1-\beta_2} -$$
$$\lambda_{H_2}B_2(1-u_1-u_2)^{\gamma_1}\gamma_1K^{\gamma_1-1}[(1-v_1-v_2)H_1]^{\gamma_2}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2}, \tag{62}$$

$$\dot{\lambda}_{H_1} = -\lambda_K A(u_1K)^{\alpha_1}v_1^{\alpha_2}\alpha_2H_1^{\alpha_2-1}(w_1H_2)^{1-\alpha_1-\alpha_2} +$$
$$\lambda_{H_1}[\delta_{H_1} - B_1(u_2K)^{\beta_1}v_2^{\beta_2}\beta_2H_1^{\beta_2-1}(w_2H_2)^{1-\beta_1-\beta_2}] -$$
$$\lambda_{H_2}B_2[(1-u_1-u_2)K]^{\gamma_1}(1-v_1-v_2)^{\gamma_2}\gamma_2H_1^{\gamma_2-1}[(1-w_1-w_2)H_2]^{1-\gamma_1-\gamma_2}, \tag{63}$$

$$\dot{\lambda}_{H_2} = -\lambda_K A(u_1K)^{\alpha_1}(v_1H_1)^{\alpha_2}w_1^{1-\alpha_1-\alpha_2}(1-\alpha_1-\alpha_2)H_2^{-\alpha_1-\alpha_2} -$$
$$\lambda_{H_1}B_1(u_2K)^{\beta_1}(v_2H_1)^{\beta_2}w_2^{1-\beta_1-\beta_2}(1-\beta_1-\beta_2)H_2^{-\beta_1-\beta_2} +$$
$$\lambda_{H_2}\{\delta_{H_2} - B_2[(1-u_1-u_2)K]^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}(1-w_1-w_2)^{1-\gamma_1-\gamma_2}(1-\gamma_1-\gamma_2)H_2^{-\gamma_1-\gamma_2}\}. \tag{64}$$

The transversality conditions are

$$\lim_{t\to\infty}\lambda_K K = 0, \tag{65}$$

$$\lim_{t\to\infty} \lambda_{H_1} H_1 = 0, \tag{66}$$

$$\lim_{t\to\infty} \lambda_{H_2} H_2 = 0. \tag{67}$$

These transversality conditions are equivalent to

$$g^*(1-\theta) < \rho, \tag{68}$$

where $g^*$ is the steady-state growth rate of the economy. We again do not consider corner solutions. From the first-order conditions it follows that

$$-\theta \frac{\dot{C}}{C} - \rho = \frac{\dot{\lambda}_K}{\lambda_K}, \tag{69}$$

$$\frac{\alpha_1}{\alpha_2}\frac{v_1}{u_1} = \frac{\beta_1}{\beta_2}\frac{v_2}{u_2} = \frac{\gamma_1}{\gamma_2}\frac{1-v_1-v_2}{1-u_1-u_2}, \tag{70}$$

$$\frac{\alpha_1}{1-\alpha_1-\alpha_2}\frac{w_1}{u_1} = \frac{\beta_1}{1-\beta_1-\beta_2}\frac{w_2}{u_2} = \frac{\gamma_1}{1-\gamma_1-\gamma_2}\frac{1-w_1-w_2}{1-u_1-u_2}, \tag{71}$$

$$\dot{\lambda}_K = \lambda_K[\delta_K - A\alpha_1(u_1K)^{\alpha_1-1}(v_1H_1)^{\alpha_2}(w_1H_2)^{1-\alpha_1-\alpha_2}], \tag{72}$$

$$\dot{\lambda}_{H_1} = \lambda_{H_1}[\delta_{H_1} - B_1(u_2K)^{\beta_1}(v_2H_1)^{\beta_2-1}\beta_2(w_2H_2)^{1-\beta_1-\beta_2}], \tag{73}$$

$$\dot{\lambda}_{H_2} = \lambda_{H_2}\{\delta_{H_2} - B_2[(1-u_1-u_2)K]^{\gamma_1}[(1-v_1-v_2)H_1]^{\gamma_2}(1-w_1-w_2)^{-\gamma_1-\gamma_2}(1-\gamma_1-\gamma_2)H_2^{-\gamma_1-\gamma_2}\}. \tag{74}$$

The equations of motion for various types of capital imply that extensive variables (output, all types of capital, and consumption) grow at the same rate in a steady state. This is the steady-state growth rate of the economy ($g^*$), which we intend to find analytically. Intensive variables ($u_1$, $v_1$, $w_1$, $u_2$, $v_2$, and $w_2$) must be constant in a steady state. The first-order conditions imply that $\lambda_K$, $\lambda_{H_1}$, and $\lambda_{H_2}$ grow at the same rate in a steady state. This rate is related to $g^*$ by equation (69). Thus, the steady-state growth rate of the economy can be derived from equations (72), (73), and (74). This growth rate can be expressed analytically if we assume identical depreciation rates for the three types of capital. We make use of the fact that the steady-state growth rates of co-state variables $\lambda$ are the same. The problem is now in the form

$$\mathcal{X}k^{\alpha_1-1}h^{\alpha_2} = \mathcal{Y}k^{\beta_1}h^{\beta_2-1} = \mathcal{Z}k^{\gamma_1}h^{\gamma_2}, \tag{75}$$

where

$$k = \frac{u_1K}{w_1H_2}, \tag{76}$$

$$h = \frac{v_1 H_1}{w_1 H_2}. \tag{77}$$

Equations (70) and (71) can help us determine parameters $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. It holds that

$$\mathcal{X} = A\alpha_1, \tag{78}$$

$$\mathcal{Y} = B_1\beta_2\left(\frac{\alpha_1/\alpha_2}{\beta_1/\beta_2}\right)^{\beta_2-1}\left[\frac{\alpha_1/(1-\alpha_1-\alpha_2)}{\beta_1/(1-\beta_1-\beta_2)}\right]^{1-\beta_1-\beta_2}, \tag{79}$$

$$\mathcal{Z} = B_2(1-\gamma_1-\gamma_2)\left(\frac{\alpha_1/\alpha_2}{\gamma_1/\gamma_2}\right)^{\gamma_2}\left[\frac{\alpha_1/(1-\alpha_1-\alpha_2)}{\gamma_1/(1-\gamma_1-\gamma_2)}\right]^{-\gamma_1-\gamma_2}. \tag{80}$$

Equation (75) basically represents a system of two non-linear equations for two variables ($k$ and $h$). Luckily, this system can be solved analytically, and its solution is given by

$$k = \left[\mathcal{X}^{\beta_2-1-\gamma_2}\mathcal{Y}^{\frac{(\alpha_1-\gamma_1-1)(\gamma_2+1-\beta_2)-1}{\beta_1-\gamma_1}}\mathcal{Z}^{\frac{(\beta_1+1-\alpha_1)(\gamma_2+1-\beta_2)+1}{\beta_1-\gamma_1}}\right]^{\xi}, \tag{81}$$

$$h = (\mathcal{X}^{\gamma_1-\beta_1}\mathcal{Y}^{\alpha_1-\gamma_1-1}\mathcal{Z}^{\beta_1+1-\alpha_1})^{\xi}, \tag{82}$$

where

$$\xi = \frac{1}{(1-\beta_2+\alpha_2)(\beta_1-\gamma_1)-(1+\beta_1-\alpha_1)(1+\gamma_2-\beta_2)}. \tag{83}$$

The common steady-state growth rate of parameters $\lambda$ is determined by $k$ and $h$. The steady-state growth rate of the economy, $g^*$, then follows from equation (69) since consumption must grow at $g^*$ in a steady state. Thus we have derived a recipe for the steady-state growth rate of the economy.

# 5 Conclusion

This paper examines three natural extensions of the Uzawa-Lucas model. The first model illustrates that the long-run growth rate critically depends on the productivity of the tertiary education sector. The second model (the Rebelo model) can also be solved analytically; we have shown that the steady-state growth rate depends positively on technological parameters and negatively on depreciation rates of human and physical capital. Additionally, we have derived formulae describing the steady-state growth rate in a natural three sector extension of the Rebelo model. The given model turned out to be relatively complicated, but we still managed to find its analytical solution. In economics, the class of models which can be solved analytically is limited. The present paper has contributed to the economic knowledge by solving three tractable models which are in some sense standard. Future research can focus on plausible extensions of these models. Alternatively, transitional dynamics (convergence towards a steady state) in these endogenous growth models can be examined.

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