On extended versions of the Solow-Swan model

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Abstract

This paper considers extensions of the neoclassical growth model with one, two, and three types of capital. The models are solved in their log-linear approximations. The dynamics of the models involve one or two negative eigenvalues. A combination of the neoclassical model with technological diffusion may lead to a nonmonotonic behavior of output. In an open-economy setup, the speed of convergence is sensitive to the specification of capital inflows. Two-capital models may result in an empirically plausible imbalance effect between human and physical capital.

Keywords: Convergence; Diffusion of technology; Eigenvalues; Imbalance effect; Neoclassical growth

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1 Introduction

The neoclassical growth model has been developed in the classic contributions of Solow (1956) and Swan (1956). This model assumes diminishing returns to capital. The model is exogenous in the sense that the long-run growth rate is determined by an exogenously given technological advance. The model implies conditional convergence: economies grow faster the further they are below their steady states. The dynamics of the Solow-Swan model are extensively discussed in Barro and Sala-i-Martin (1995, Chapter 1).

The present paper considers several extensions of the Solow-Swan model. The given extensions represent original contributions of this paper. The paper first examines the speed of convergence in a model with one type of capital and a general production function. It is shown that the convergence coefficient is independent of the saving rate only for a Cobb-Douglas production function. The paper then investigates simple specifications in which depreciation and the saving rate depend positively on capital per effective worker. The dependence of depreciation on capital increases the convergence coefficient, while the dependence of the saving rate on capital slows down convergence.

The paper then introduces technological diffusion in the neoclassical growth model. The model can be solved in its log-linear approximation. The model exhibits two negative eigenvalues (each eigenvalue corresponding to one source of convergence - diminishing returns or technological diffusion), and its dynamics are richer than the dynamics of the basic Solow-Swan model. An important aspect of the model with technological diffusion is the possibility of a nonmonotonic behavior of output if capital per effective worker is high and technology is low initially.

The paper also considers an open-economy version of the Solow-Swan model in which foreign capital inflows are a positive function of rates of return on capital. As expected, an open economy tends to converge faster than a closed economy. The speed of convergence of an open economy is, nevertheless, extremely sensitive to a functional specification of capital inflows.

In addition, the paper focuses on models with two types of capital (physical capital and human capital). In the presence of different depreciation rates, the model's dynamics of output depend on two negative eigenvalues. On the other hand, if there are identical depreciation rates, output's dynamics involve only one negative eigenvalue. If production depends only on physical capital and the saving rate depends positively on human capital, the growth rate of output depends positively on human capital and negatively on physical capital. This is an empirically plausible imbalance effect between human and physical capital. Additionally, the paper solves a more general framework with both types of capital entering into production and saving rates depending on ratios of both types of capital. The model's dynamics involve two negative eigenvalues. Depending on the parameters of the model, there is an imbalance effect between human and physical capital. The paper is ended up with a variant of the Solow-Swan model with three types of capital. The model can be solved analytically in its log-linear approximation. For each capital type the dynamics involve two negative eigenvalues, while there is still just one negative eigenvalue for output.

2 One type of capital

2.1 The basic model

The production function is assumed to exhibit constant returns to scale in capital and effective labor:

$$Y = F(K, AL), \tag{1}$$

where Y is output, K is capital, A is the level of technology, and L is labor. Output per effective worker, $y = Y/(AL)$, is a concave and increasing function of capital per effective worker, $k = K/(AL)$:

$$y = F[K/(AL), 1] = f(k) \tag{2}$$

The equation of motion for capital is

$$\dot{K} = sF(K, AL) - \delta K, \tag{3}$$

which can be rewritten in the intensive form

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n + x), \tag{4}$$

where s is the exogenously given saving rate, δ is the depreciation rate of capital, n is the population growth rate, and x is the growth rate of technology. The steady-state level of capital per effective worker, k^*, satisfies

$$s\frac{f(k^*)}{k^*} = \delta + n + x. \tag{5}$$

Using identity $k = e^{\ln k}$, the equation of motion for k can be log-linearized around the steady state:

$$\frac{\dot{k}}{k} = \frac{d\ln(k/k^*)}{dt} = s[f'(k^*) - f(k^*)/k^*]\ln(k/k^*), \tag{6}$$

which can be written as

$$\frac{\dot{k}}{k} = -\beta\ln(k/k^*), \tag{7}$$

where

$$\beta = s[f(k^*)/k^* - f'(k^*)] = (\delta + n + x)[1 - f'(k^*)k^*/f(k^*)] \tag{8}$$

is the convergence coefficient. If the production function is Cobb-Douglas:

$$y = f(k) = k^\alpha, \tag{9}$$

where $0 < \alpha < 1$ is the capital share, the convergence coefficient is (see also Barro and Sala-i-Martin, 1995, Chapter 1)

$$\beta = (1 - \alpha)(\delta + n + x). \tag{10}$$

It is straightforward to show that the convergence coefficient is independent of the saving rate only for a Cobb-Douglas production function. Coefficient β does not depend on s if and only if $f'(k^*)$ is a multiple of $f(k^*)/k^*$:

$$f'(k^*) = \nu f(k^*)/k^*, \tag{11}$$

where ν is a parameter independent of the saving rate. This can be integrated as

$$\ln f(k^*) = \nu\ln k^* + C, \tag{12}$$

where C is a constant of integration. This equation can be rewritten as

$$f(k^*) = e^C k^{*\nu}, \tag{13}$$

which is the Cobb-Douglas production function.

2.2 Capital-dependent depreciation

In the real world, the depreciation of capital depends on the type of capital. For example, structures depreciate at a lower rate than equipment. If equipment investment prevails over investment in structures, the average depreciation rate depends positively on capital. It is worth examining the convergence effects of the depreciation rate depending on capital. Let

$$\delta = b_1 + b_2 k^{\phi}, \tag{14}$$

where b_1, b_2, and ϕ are positive parameters. Let the production function be Cobb-Douglas:

$$y = k^{\alpha} \tag{15}$$

The equation of motion for capital is

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (b_1 + x + n) - b_2 k^{\phi}. \tag{16}$$

The steady-state level of capital per effective worker satisfies:

$$sk^{*\alpha-1} - b_2 k^{*\phi} = b_1 + x + n. \tag{17}$$

Thus the presence of the term $b_2 k^{*\phi}$ decreases k^*. The speed of convergence is

$$\beta = (1-\alpha)sk^{*\alpha-1} + b_2 \phi k^{*\phi}, \tag{18}$$

$$\beta = (1-\alpha)(b_1 + x + n) + b_2 k^{*\phi}(1 - \alpha + \phi). \tag{19}$$

The term $b_2 k^{*\phi}$ unambiguously increases the convergence coefficient.

2.3 A capital-dependent saving rate

Mankiw (2000, p. 88) provides empirical evidence concerning the association between real economic activity and investment rates. This association is clearly positive. One explanation is causality from investment to output. This explanation is consistent with the prediction of the basic Solow model for the steady-state level of output per effective worker. Nevertheless, reverse causality is also possible. Financial intermediaries are more developed in rich countries. The developed financial intermediaries facilitate the transfer of funds from savers to borrowers.

Savers have safer and higher returns, which encourages saving. In the present context, we model this phenomenon by the saving rate depending positively on capital per effective worker:

$$s = a_1 + a_2 k^{\psi}, \tag{20}$$

where a_1, a_2, and ψ are positive parameters. The production function is again Cobb-Douglas; the depreciation rate is now assumed to be independent of capital. The parameter ψ is assumed to be sufficiently small, so that $\alpha + \psi < 1$ (this condition is necessary to exclude endogenous growth). The equation of motion for capital is

$$\frac{\dot{k}}{k} = a_1 k^{\alpha-1} + a_2 k^{\alpha+\psi-1} - (\delta + x + n). \tag{21}$$

The steady-state level of capital per effective worker satisfies

$$a_1 k^{*\alpha-1} + a_2 k^{*\alpha+\psi-1} = \delta + x + n. \tag{22}$$

The presence of the term with a_2 increases k^*. The convergence coefficient satisfies

$$\beta = (1-\alpha) a_1 k^{*\alpha-1} + (1-\alpha-\psi) a_2 k^{*\alpha+\psi-1}, \tag{23}$$

$$\beta = (1-\alpha)(\delta + x + n) - \psi a_2 k^{*\alpha+\psi-1}. \tag{24}$$

The term with a_2 slows down convergence. Intuitively, the saving rate depending positively on capital increases the effective capital share in the equation of motion for capital.

2.4 Technological diffusion

Let the aggregate production function be Cobb-Douglas:

$$Y = K^{\alpha}(AL)^{1-\alpha} \tag{25}$$

Let the technological parameter increase faster the further it is below the level of technology in the leading country, A_L:

$$x = \frac{\dot{A}}{A} = g + \lambda \frac{A_L - A}{A}, \tag{26}$$

where g and λ are positive parameters. The term g reflects domestic forces of the technical change (domestic research and development). The second term on the right-hand side corresponds to technological diffusion from the leading country. Nelson and Phelps (1966) proposed a similar expression. Barro and Sala-i-Martin (1997) provide microeconomic foundations for the Nelson and Phelps framework. In the context of Nelson and Phelps' paper, λ was a positive function of the domestic human-capital intensity. In the present paper we abstract from this consideration; λ is a positive constant. We assume that the growth rate of A_L is constant and given by

$$\frac{\dot{A}_L}{A_L} = g. \tag{27}$$

Let $a = A/A_L$ denote the intensive form of the domestic technological parameter. The equation of motion for a is

$$\frac{\dot{a}}{a} = \frac{\dot{A}}{A} - g = \lambda(1/a - 1) \approx -\lambda \ln a. \tag{28}$$

The steady-state level of a is $a^* = 1$. In terms of variables per effective worker, the production function is

$$y = Y/(AL) = [K/(AL)]^\alpha = k^\alpha. \tag{29}$$

The equation of motion for aggregate capital is

$$\dot{K} = sY - \delta K = sK^\alpha (AL)^{1-\alpha} - \delta K. \tag{30}$$

The equation of motion for capital per effective worker is

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - x - n = sk^{\alpha-1} - (\delta + g + n) - \lambda(1/a - 1). \tag{31}$$

The steady-state level of k satisfies

$$sk^{*\alpha-1} = \delta + g + n. \tag{32}$$

The equation of motion for k can be log-linearized around the steady state:

$$\frac{\dot{k}}{k} = (\alpha - 1)(\delta + g + n)\ln(k/k^*) + \lambda \ln a \tag{33}$$

The Jacobian matrix of the system of equations for k and a is

$$\begin{pmatrix} (\alpha - 1)(\delta + g + n) & \lambda \\ 0 & -\lambda \end{pmatrix}$$

The Jacobian matrix has two negative eigenvalues:

$$\epsilon_1 = -(1-\alpha)(\delta + n + g), \tag{34}$$

$$\epsilon_2 = -\lambda \tag{35}$$

The first eigenvalue corresponds to the standard Solow model; this eigenvalue reflects diminising returns to capital. The second eigenvalue corresponds to technological diffusion. The equations for k and a are

$$\ln(k/k^*) = \nu_1 e^{-(1-\alpha)(\delta+g+n)t} + \nu_2 e^{-\lambda t}, \tag{36}$$

$$\ln a = \nu_2[-1 + (1-\alpha)(\delta + n + g)/\lambda]e^{-\lambda t}. \tag{37}$$

Coefficients in the equation for a relative to coefficients in the equation for k are determined by components of eigenvectors corresponding to the given eigenvalues. Coefficients ν_1 and ν_2 can be expressed from initial conditions:

$$\nu_1 = \ln(k(0)/k^*) - \frac{\ln a(0)}{(1-\alpha)(\delta+n+g)/\lambda - 1} \tag{38}$$

$$\nu_2 = \frac{\ln a(0)}{(1-\alpha)(\delta+n+g)/\lambda - 1}. \tag{39}$$

Output per capita, y_c, satisfies

$$y_c = yA = yaA_L = aA_L k^\alpha. \tag{40}$$

In the leading country, ouptut per capita, $y_{c,L}$, is given by

$$y_{c,L} = A_L k^{*\alpha}. \tag{41}$$

This is the steady-state target, y_c^*, for the converging country. It holds that

$$\ln(y_c/y_c^*) = \ln a + \alpha \ln(k/k^*), \tag{42}$$

$$\ln(y_c/y_c^*) = \nu_{y1} e^{-(1-\alpha)(\delta+n+g)t} + \nu_{y2} e^{-\lambda t}, \tag{43}$$

where

$$\nu_{y1} = \alpha\Big[\ln(k(0)/k^*) - \frac{\ln a(0)}{(1-\alpha)(\delta+n+g)/\lambda - 1}\Big], \tag{44}$$

$$\nu_{y2} = \ln a(0)\Big(1 + \frac{\alpha}{(1-\alpha)(\delta+n+g)/\lambda - 1}\Big). \tag{45}$$

The dynamics with two negative eigenvalues are richer than the dynamics of the basic Solow model. Elsewhere I have developed other models with two negative eigenvalues. Duczynski (2002a) examines an open-economy neoclassical growth model with two types of capital and different adjustment costs. Due to the presence of two negative eigenvalues, the evolution of output depends on the initial ratio of human to physical capital. If adjustment costs are larger for human than for physical capital, the growth of output depends positively on the ratio of human to physical capital. This is empirically plausible [see, for example, Barro and Sala-i-Martin (1995, Chapter 12), where the growth rate of output is found to be positively dependent on human capital for a given output level]. In Duczynski (2002b) I introduce technological diffusion in the Ramsey model. The model is somewhat more complicated than the Solow model with technological diffusion because saving decisions are endogenized in the Ramsey model. Duczynski (2003) solves an open-economy model with one type of capital, adjustment costs, and technological diffusion. Again, the growth rate of output depends on initial conditions (initial conditions for capital and technology). The speed of convergence for output is fast in initially undercapitalized economies, which is probably the case of transition economies. A number of these economies import foreign capital and grow rapidly.

In the present model, the speed of convergence for output is given by

$$\beta = -\frac{\frac{d\ln(y_c/y_c^*)}{dt}}{\ln(y_c/y_c^*)} = \frac{(1-\alpha)(\delta+n+g)\nu_{y1}e^{-(1-\alpha)(\delta+n+g)t} + \lambda\nu_{y2}e^{-\lambda t}}{\nu_{y1}e^{-(1-\alpha)(\delta+n+g)t} + \nu_{y2}e^{-\lambda t}}. \quad (46)$$

The convergence coefficient expresses how the growth rate of output changes with the distance of the economy from the steady state. The convergence coefficient is a weighted average of absolute values of the eigenvalues of the model.

It can easily be shown that y_c/y_c^* can be nonmonotonic over time. Let us consider the following specification: $g = 0.02$, $\delta = 0.05$, $n = 0.01$, $\alpha = 0.3$, $\lambda = 0.01$, $a(0) = 0.5$, and $k(0)/k^* = 1.5$. For this specification, the equation for output is approximately

$$\ln(y_c/y_c^*) = 0.167e^{-0.056t} - 0.738e^{-0.01t}. \quad (47)$$

Output relative to its steady state initially decreases but asymptotically increases. In this situation, diminishing returns to capital work initially in the opposite direction than technological diffusion. This is a significant difference from the standard Solow model, where the evolution of capital and output is monotonic. In Duczynski (2002b) I examine the combination of technological diffusion with the Ramsey model. I observe that the evolution of output can also be nonmonotonic in the Ramsey model with technological diffusion.

2.5 An open economy

Let us consider an open economy which can import foreign capital. Let the production function be Cobb-Douglas. Foreign-capital inflows are assumed to depend positively on the gross return on domestic capital, $\alpha k^{\alpha-1}$, and negatively on the gross return on the world credit market, ρ. The trade-balance deficit equals $\nu(\alpha k^{\alpha-1}-\rho)^m$, where ν and m are positive parameters. We assume that returns on foreign capital are automatically reinvested. The equation of motion for capital is

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta+n+x) + \nu\frac{(\alpha k^{\alpha-1}-\rho)^m}{k}. \tag{48}$$

The steady-state capital per effective worker satisfies

$$sk^{*\alpha-1} = \delta+n+x, \tag{49}$$

$$\alpha k^{*\alpha-1} = \rho. \tag{50}$$

From this it holds for the parameters of the model that

$$\rho = \frac{\delta+n+x}{s}\alpha. \tag{51}$$

The equation of motion for capital can be log-linearized around the steady state:

$$\frac{\dot{k}}{k} = -\beta\ln(k/k^*), \tag{52}$$

where the convergence coefficient satisfies

$$\beta = (1-\alpha)(x+n+\delta) + \nu(1-\alpha)\frac{\rho}{k^*}m(\alpha k^{*\alpha-1}-\rho)^{m-1}. \tag{53}$$

For $m > 1$, the speed of convergence is the same as for a closed economy:

$$\beta = (1-\alpha)(x+n+\delta) \tag{54}$$

For $m < 1$, the speed of converence tends to infinity if the economy is below its steady state. Thus we observe an extremely high sensitivity of the speed of convergence to the functional specification of capital inflows depending on rate-of-return differentials. The medium case of $m = 1$ leads to

$$\beta = (1-\alpha)(\delta + n + x) + \nu(1-\alpha)\frac{\rho}{k^*}. \tag{55}$$

This speed of convergence is higher than in the case of a closed economy. It is of certain interest to quantify the difference of this speed of convergence from the closed-economy speed of convergence. Let $s = 0.5$, $\alpha = 0.75$ (capital is a composite of human and physical capital), $\delta = 0.05$, $n = 0.01$, and $x = 0.02$. For this specification, $\rho = 0.12$, $k^* = 1525.88$, and $y^* = 244.14$. Let the initial capital stock be at 1/2 of the steady state. The rate-of-return differential is then 0.023. The parameter ν is chosen such that the initial trade-balance deficit is reasonably low. If, for example, $\nu = 200$, the initial trade-balance deficit is approximately 1.9% of output. Then the convergence coefficient is 0.024, as compared with the closed-economy value of 0.02. Thus the open-economy consideration in the case of a linear specification of capital inflows does not tend to increase the speed of convergence substantially as long as the trade-balance deficit is not a very high portion of output.

3 Two types of capital

3.1 Different depreciation rates

Mankiw, Romer, and Weil (1992) consider a neoclassical growth model with physical and human capital. We extend their model by assuming different depreciation rates for physical and human capital. The equations of motion for physical and human capital per effective worker are

$$\dot{k} = s_k k^\alpha h^\eta - (\delta_k + x + n)k, \tag{56}$$

$$\dot{h} = s_h k^\alpha h^\eta - (\delta_h + x + n)h, \tag{57}$$

where s_k and s_h, respectively, are saving rates for physical and human capital, respectively, and δ_k and δ_h, respectively, are depreciation rates for physical and human capital, respectively. Diminishing returns apply to broad capital, i.e., $\alpha + \eta < 1$. The equations of motion can be log-linearized around the steady state:

$$\frac{\dot{k}}{k} = (\alpha - 1)(\delta_k + x + n)\ln(k/k^*) + \eta(\delta_k + x + n)\ln(h/h^*), \tag{58}$$

$$\frac{\dot{h}}{h} = \alpha(\delta_h + x + n)\ln(k/k^*) + (\eta - 1)(\delta_h + x + n)\ln(h/h^*). \tag{59}$$

The Jacobian matrix has two negative eigenvalues:

$$\lambda_{1,2} = \frac{\mathcal{A} + \mathcal{D} \pm \sqrt{(\mathcal{A} + \mathcal{D})^2 - 4(\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C})}}{2}, \tag{60}$$

where

$$\mathcal{A} = (\alpha - 1)(\delta_k + x + n), \tag{61}$$

$$\mathcal{B} = \eta(\delta_k + x + n), \tag{62}$$

$$\mathcal{C} = \alpha(\delta_h + x + n), \tag{63}$$

$$\mathcal{D} = (\eta - 1)(\delta_h + x + n). \tag{64}$$

The solution to the log-linearized system takes the form:

$$\ln(k/k^*) = \nu_1 e^{\lambda_1 t} + \nu_2 e^{\lambda_2 t}, \tag{65}$$

$$\ln(h/h^*) = \nu_1 \frac{\lambda_1 - \mathcal{A}}{\mathcal{B}} e^{\lambda_1 t} + \nu_2 \frac{\lambda_2 - \mathcal{A}}{\mathcal{B}} e^{\lambda_2 t}. \tag{66}$$

Coefficients in the equation for human capital are derived from coefficients in the equation for physical capital using the components of eigenvectors of the Jacobian matrix. Coefficients ν_1 and ν_2 are determined from initial conditions. The equation of output is given by

$$\ln(y/y^*) = \alpha \ln(k/k^*) + \eta \ln(h/h^*). \tag{67}$$

If there are different depreciation rates, the evolution of output is determined by the interaction of two terms with different negative eigenvalues. Thus the model belongs to the class of models with two negative eigenvalues which have been briefly mentioned in the section devoted to technological diffusion. The model's dynamics depend on initial human and physical capital and are richer than the dynamics of the basic Solow-Swan model.

The situation is simpler if there are identical depreciation rates. In this case,

$$\lambda_1 = (\alpha + \eta - 1)(\delta + n + x), \tag{68}$$

$$\lambda_2 = -(\delta + n + x), \tag{69}$$

$$\frac{\lambda_1 - \mathcal{A}}{\mathcal{B}} = 1, \tag{70}$$

$$\frac{\lambda_2 - \mathcal{A}}{\mathcal{B}} = -\frac{\alpha}{\eta}, \tag{71}$$

$$\nu_1 = \ln(k(0)/k^*) - \frac{\ln(k(0)/k^*) - \ln(h(0)/h^*)}{1 + \alpha/\eta}, \tag{72}$$

$$\nu_2 = \frac{\ln(k(0)/k^*) - \ln(h(0)/h^*)}{1 + \alpha/\eta}. \tag{73}$$

The evolution of output is given by

$$\ln(y/y^*) = \nu_1(\alpha + \eta)e^{\lambda_1 t}. \tag{74}$$

The term with $e^{\lambda_2 t}$ disappears. Therefore, Mankiw, Romer, and Weil (1992) correctly presented the evolution of output depending on one negative eigenvalue [negative convergence coefficient equal to $-(1 - \alpha - \eta)(x + n + \delta)$].

3.2 A human-capital dependent saving rate

This subsection considers a model in which the saving rate for physical capital depends on human capital:

$$s_k = bh^\epsilon, \tag{75}$$

where b and ϵ are positive parameters. A higher level of human capital may reflect better knowledge about investment opportunities. People with human capital may be more able to carry out productive investment. Alternatively, a high level of human capital in the economy is connected with high rates of return on physical capital (human and physical capital are complements in production), which encourages investment in physical capital. Despite this plausible consideration, we make the analysis simple by assuming that only physical capital enters into production: $y = k^\alpha$. The saving rate for human capital is assumed to be constant. The equations of motion for physical and human capital per effective worker are

$$\frac{\dot{k}}{k} = bh^\epsilon k^{\alpha-1} - (\delta + n + x), \tag{76}$$

$$\frac{\dot{h}}{h} = s_h k^\alpha h^{-1} - (\delta + n + x). \tag{77}$$

These equations can be log-linearized around the steady state:

$$\frac{\dot{k}}{k} = (\alpha - 1)(\delta + n + x)\ln(k/k^*) + \epsilon(\delta + n + x)\ln(h/h^*), \tag{78}$$

$$\frac{\dot{h}}{h} = \alpha(\delta + n + x)\ln(k/k^*) - (\delta + n + x)\ln(h/h^*). \tag{79}$$

The eigenvalues of the Jacobian matrix are given by

$$\lambda_{1,2} = \frac{(\alpha - 2)(\delta + n + x) \pm \sqrt{(2-\alpha)^2(\delta + n + x)^2 - 4[(1-\alpha) - \alpha\epsilon](\delta + n + x)^2}}{2}. \tag{80}$$

We assume that ϵ is sufficiently small ($\epsilon\alpha < 1 - \alpha$), so that both eigenvalues are negative. It holds that

$$\lambda_1 \geq -(1-\alpha)(\delta + n + x), \tag{81}$$

$$\lambda_2 \leq -(\delta + n + x), \tag{82}$$

where equalities hold for $\epsilon = 0$. The solutions to the log-linearized system take the form:

$$\ln(k/k^*) = \nu_1 e^{\lambda_1 t} + \nu_2 e^{\lambda_2 t}, \tag{83}$$

$$\ln(h/h^*) = \nu_1 \mathcal{A}_1 e^{\lambda_1 t} + \nu_2 \mathcal{A}_2 e^{\lambda_2 t}, \tag{84}$$

where

$$\mathcal{A}_1 = \frac{\alpha(\delta + n + x)}{\delta + n + x + \lambda_1} > 0 \tag{85}$$

and

$$\mathcal{A}_2 = \frac{\alpha(\delta + n + x)}{\delta + n + x + \lambda_2} < 0 \tag{86}$$

are derived from eigenvectors of the Jacobian matrix. Coefficients ν_1 and ν_2 depend on initial conditions:

$$\nu_1 = \frac{\mathcal{A}_2 \ln(k(0)/k^*) - \ln(h(0)/h^*)}{\mathcal{A}_2 - \mathcal{A}_1} \tag{87}$$

$$\nu_2 = \frac{\ln(h(0)/h^*) - \mathcal{A}_1 \ln(k(0)/k^*)}{\mathcal{A}_2 - \mathcal{A}_1} \tag{88}$$

If $\epsilon \to 0$, then $\mathcal{A}_2 \to -\infty$ and $\nu_2 \to 0$. In this case the dynamics of the system are described by one eigenvalue.

The evolution of output is given by

$$\ln(y/y^*) = \alpha\nu_1 e^{\lambda_1 t} + \alpha\nu_2 e^{\lambda_2 t}. \tag{89}$$

The initial growth rate of output per effective worker satisfies

$$g_y = \alpha\nu_1\lambda_1 + \alpha\nu_2\lambda_2, \tag{90}$$

$$g_y = -\alpha\Big[|\lambda_1|\frac{\mathcal{A}_2\ln(k(0)/k^*) - \ln(h(0)/h^*)}{\mathcal{A}_2 - \mathcal{A}_1} + |\lambda_2|\frac{\ln(h(0)/h^*) - \mathcal{A}_1\ln(k(0)/k^*)}{\mathcal{A}_2 - \mathcal{A}_1}\Big]. \tag{91}$$

Because $|\lambda_1| < |\lambda_2|$ and $\mathcal{A}_2 - \mathcal{A}_1 < 0$, the initial growth rate of output depends positively on initial human capital and negatively on initial physical capital. This is an empirically plausible imbalance effect between human and physical capital. Economies which suffered a sudden decline of physical capital (such as economies after wars) really tended to grow rapidly. On the other hand, economies that lost human capital (such as economies after the plague in the Middle Ages) did not grow fast. Barro (1991) and Barro and Sala-i-Martin (1995, Chapter 12) show that in a cross-section of countries the growth rate of output depends positively on human capital for a given output level. In Duczynski (2002a) I show that an empirically plausible imbalance effect between human and physical capital can be derived in a two-capital model with adjustment costs if these adjustment costs are larger for human than for physical capital.

3.3 Saving rates depending on rates of return

We now assume that both types of capital enter into a Cobb-Douglas production function. The rate of return on physical capital depends positively on the ratio of human to physical capital, while the rate of return on human capital depends positively on the ratio of physical to human capital. We assume that the saving rates depend on the ratios of both types of capital:

$$s_k = s_{k0}(h/k)^{\epsilon_1}, \tag{92}$$

$$s_h = s_{h0}(k/h)^{\epsilon_2}, \tag{93}$$

where ϵ_1 and ϵ_2 are positive parameters. The equations of motion for physical and human capital are

$$\frac{\dot{k}}{k} = s_{k0}(h/k)^{\epsilon_1}k^{\alpha-1}h^{\eta} - (\delta + n + x), \tag{94}$$

$$\frac{\dot{h}}{h} = s_{h0}(k/h)^{\epsilon_2}k^{\alpha}h^{\eta-1} - (\delta + n + x). \tag{95}$$

After a log-linearization around the steady state we obtain:

$$\frac{\dot{k}}{k} = (\alpha - 1 - \epsilon_1)(\delta + n + x)\ln(k/k^*) + (\epsilon_1 + \eta)(\delta + n + x)\ln(h/h^*), \quad (96)$$

$$\frac{\dot{h}}{h} = (\alpha + \epsilon_2)(\delta + n + x)\ln(k/k^*) + (\eta - 1 - \epsilon_2)(\delta + n + x)\ln(h/h^*). \quad (97)$$

The eigenvalues of the Jacobian matrix are given by

$$\lambda_1 = -(1 - \alpha - \eta)(\delta + n + x), \quad (98)$$

$$\lambda_2 = -(1 + \epsilon_1 + \epsilon_2)(\delta + n + x). \quad (99)$$

The solution to the log-linearized system takes the form:

$$\ln(k/k^*) = \nu_1 e^{\lambda_1 t} + \nu_2 e^{\lambda_2 t}, \quad (100)$$

$$\ln(h/h^*) = \nu_1 e^{\lambda_1 t} - \frac{\alpha + \epsilon_2}{\eta + \epsilon_1}\nu_2 e^{\lambda_2 t}, \quad (101)$$

where ν_1 and ν_2 are determined by initial conditions:

$$\nu_1 = \ln(h(0)/h^*)\frac{\eta + \epsilon_1}{\alpha + \eta + \epsilon_1 + \epsilon_2} + \ln(k(0)/k^*)\frac{\alpha + \epsilon_2}{\alpha + \eta + \epsilon_1 + \epsilon_2}, \quad (102)$$

$$\nu_2 = [\ln(k(0)/k^*) - \ln(h(0)/h^*)]\frac{\eta + \epsilon_1}{\alpha + \eta + \epsilon_1 + \epsilon_2}. \quad (103)$$

The evolution of output is

$$\ln(y/y^*) = \alpha\ln(k/k^*) + \eta\ln(h/h^*) = (\alpha + \eta)\nu_1 e^{\lambda_1 t} + \frac{\alpha\epsilon_1 - \eta\epsilon_2}{\eta + \epsilon_1}\nu_2 e^{\lambda_2 t}. \quad (104)$$

[If $\epsilon_1 = \epsilon_2 = 0$, output depends only on the term with the eigenvalue λ_1. This is what we derived in the first subsection of this section and what Mankiw, Romer, and Weil (1992) used in their paper.] The initial growth rate of output per effective worker satisfies

$$g_y = \frac{\dot{y}}{y} = \frac{\delta + n + x}{\alpha + \eta + \epsilon_1 + \epsilon_2}[\kappa_h \ln(h(0)/h^*) + \kappa_k \ln(k(0)/k^*)], \quad (105)$$

where

$$\kappa_h = -(\eta + \epsilon_1)(\alpha + \eta)(1 - \alpha - \eta) + (\alpha\epsilon_1 - \eta\epsilon_2)(1 + \epsilon_1 + \epsilon_2), \quad (106)$$

$$\kappa_k = -(\alpha + \eta)(1 - \alpha - \eta)(\alpha + \epsilon_2) + (\eta\epsilon_2 - \alpha\epsilon_1)(1 + \epsilon_1 + \epsilon_2). \quad (107)$$

The growth rate of output per effective worker depends on each type of capital through two terms. The first term depends on diminishing returns to broad capital. This term contains $1-\alpha-\eta$ and disappears if there are constant returns on capital. The second term depends on $\alpha\epsilon_1 - \eta\epsilon_2$. If $\alpha\epsilon_1 > \eta\epsilon_2$, the growth of output depends positively on human capital and negatively on physical capital (if we abstract from the first term). This result is quite intuitive. If ϵ_1 is relatively high, the positive effect of a high ratio of human to physical capital on s_k prevails over the negative effect on s_h. The resulting imbalance effect between human and physical capital is then empirically plausible.

4 Three types of capital

Production depends on three types of capital:

$$y = k_1^{\alpha_1} k_2^{\alpha_2} k_3^{\alpha_3} \tag{108}$$

The equations of motion for each capital type are:

$$\frac{\dot{k}_1}{k_1} = s_1 k_1^{\alpha_1 - 1} k_2^{\alpha_2} k_3^{\alpha_3} - (\delta + n + x), \tag{109}$$

$$\frac{\dot{k}_2}{k_2} = s_2 k_1^{\alpha_1} k_2^{\alpha_2 - 1} k_3^{\alpha_3} - (\delta + n + x), \tag{110}$$

$$\frac{\dot{k}_3}{k_3} = s_3 k_1^{\alpha_1} k_2^{\alpha_2} k_3^{\alpha_3 - 1} - (\delta + n + x). \tag{111}$$

These equations can be log-linearized around the steady state:

$$\frac{\dot{k}_1}{k_1} = (\alpha_1 - 1)(\delta+n+x)\ln(k_1/k_1^*) + \alpha_2(\delta+n+x)\ln(k_2/k_2^*) + \alpha_3(\delta+n+x)\ln(k_3/k_3^*), \tag{112}$$

$$\frac{\dot{k}_2}{k_2} = \alpha_1(\delta+n+x)\ln(k_1/k_1^*) + (\alpha_2 - 1)(\delta+n+x)\ln(k_2/k_2^*) + \alpha_3(\delta+n+x)\ln(k_3/k_3^*), \tag{113}$$

$$\frac{\dot{k}_3}{k_3} = \alpha_1(\delta+n+x)\ln(k_1/k_1^*) + \alpha_2(\delta+n+x)\ln(k_2/k_2^*) + (\alpha_3 - 1)(\delta+n+x)\ln(k_3/k_3^*). \tag{114}$$

The Jacobian matrix has two negative eigenvalues:

$$\lambda_1 = -(1 - \alpha_1 - \alpha_2 - \alpha_3)(\delta + n + x), \tag{115}$$

$$\lambda_2 = -(\delta + n + x). \tag{116}$$

The eigenvalue λ_2 is a double eigenvalue. The solution to the log-linearized system takes the form:

$$\ln(k_1/k_1^*) = \nu_1 e^{\lambda_1 t} + \nu_2 e^{\lambda_2 t}, \tag{117}$$

$$\ln(k_2/k_2^*) = \nu_1 e^{\lambda_1 t} + \nu_3 e^{\lambda_2 t}, \tag{118}$$

$$\ln(k_3/k_3^*) = \nu_1 e^{\lambda_1 t} - \frac{\alpha_1 \nu_2 + \alpha_2 \nu_3}{\alpha_3} e^{\lambda_2 t}. \tag{119}$$

Coefficients ν_1, ν_2, and ν_3 can be determined from initial conditions. The equation for output is

$$\ln(y/y^*) = \alpha_1 \ln(k_1/k_1^*) + \alpha_2 \ln(k_2/k_2^*) + \alpha_3 \ln(k_3/k_3^*) = (\alpha_1 + \alpha_2 + \alpha_3)\nu_1 e^{\lambda_1 t}. \tag{120}$$

Similar to the simple two-capital model, the dynamics of output are given by only one negative eigenvalue.

5 Conclusion

This paper considers extensions of the Solow-Swan neoclassical growth model with one, two, and three types of capital. These models are solved analytically in their log-linear approximations. One-capital models involve one negative eigenvalue; its absolute value equals the convergence coefficient. If the Solow-Swan model is combined with technological diffusion, there are two negative eigenvalues. With technological diffusion the model's dynamics are relatively rich and the behavior of output over time may be nonmonotonic. If the Solow-Swan model is discussed in an open-economy perspective, the speed of convergence tends to be increased. If capital inflows depend linearly on rates of return, the increment in the speed of convergence is not high if trade-balance deficits are low in early stages of development. If capital inflows depend less than linearly (more than linearly, respectively) on rates of return, the increment in the speed of convergence in an open economy is infinitely large (zero, respectively).

Models with two types of capital involve two negative eigenvalues which determine the evolution of both types of capital. However, for the basic model with identical depreciation rates, the dynamics for output include only one negative eigenvalue. These models can exhibit an empirically plausible imbalance effect between human and physical capital if the saving rate for physical capital depends positively on human capital. An imbalance effect also occurs if the saving rate for physical capital depends positively on the ratio of human to physical capital, while the human-capital saving rate depends positively on the ratio of physical to human capital. The Solow-Swan model with three types of capital can also be solved analytically in its log-linear approximation. The evolution of each type of capital is given by two negative eigenvalues. In the simple version of the model, the evolution of output is determined by only one negative eigenvalue.

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