A Note on the Imbalance Effect in the Uzawa-Lucas Model

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Abstract

The Uzawa-Lucas model is believed to yield a positive dependence of the output growth on the ratio of human capital to physical capital (an empirically plausible imbalance effect). We show that the imbalance effect becomes less plausible for a low physical capital share and a low elasticity of intertemporal substitution. In particular, the model is inconsistent with empirical observations for a relatively broad range of realistic parameter specifications.

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1 Introduction

The Uzawa-Lucas model (see Uzawa, 1965, and Lucas, 1988) is one of the most important endogenous growth models. This model considers two sectors - one sector producing physical capital goods (which can be converted into consumption), and the other sector producing human capital. The production function of the physical capital sector is Cobb-Douglas in physical and human capital, while the production of human capital depends only on human capital. The Uzawa-Lucas model (without externalities) has a globally stable steady state in which consumption, output, human capital, and physical capital grow at the same rate.

Rebelo (1991) extends the Uzawa-Lucas model by assuming that physical capital is used in the production of human capital. He shows that the observed cross-country disparities in output growth rates can be explained by differences in government policy. Caballé and Santos (1993) consider a two-sector model with general linearly homogeneous production functions and demonstrate a global-convergence property. Mulligan and Sala-i-Martin (1993) examine convergence towards a steady state in two-sector models. One of their important results is that the Uzawa-Lucas model exhibits an empirically plausible imbalance effect between human and physical capital: the output growth depends positively on the ratio of human capital to physical capital. Xie (1994) shows that the Uzawa-Lucas model can exhibit multiple equilibria if there are large externalities from human capital in the physical capital sector. Benhabib and Perli (1994) and Ladrón-de-Guevara, Ortigueira, and Santos (1997) demonstrate that multiple steady states may also result from the inclusion of labor-leisure choice.

The present paper solves the log-linearized Uzawa-Lucas model. This solution reveals some characteristics of the model in the neighborhood of the steady state. Regarding narrowly defined output, the imbalance effect between human and physical capital is empirically plausible only if the physical capital share in the physical capital sector is high. If this capital share is low (such as 1/3 or 0.25), the output growth depends *negatively* on the human-physical ratio, which is implausible. We show that this important and surprising weakness is mitigated if output is defined broadly (consisting of narrow output and investments in human capital). Nevertheless, even for measured output, which is a weighted average of narrow and broad outputs, the imbalance effect tends to be unrealistic for some reasonable parameter specifications.

2 The model

There are no externalities and no population growth. The problem is

$$\max_{C,u} \int_0^\infty \ln C e^{-\rho t} dt$$

subject to

$$\dot{K} = AK^\alpha (uH)^{1-\alpha} - C - \delta K, \tag{1}$$

$$\dot{H} = B(1-u)H - \delta H, \tag{2}$$

where C is consumption, K is physical capital, H is human capital (extensive variables), u is the fraction of human capital employed in the physical capital sector (an intensive variable), A and B are fixed technological parameters, ρ is the rate of time preference, α is the physical capital share, and δ is the depreciation rate of physical and human capital. The present-value Hamiltonian is

$$\mathcal{J} = \ln C e^{-\rho t} + \lambda_K[AK^\alpha (uH)^{1-\alpha} - C - \delta K] + \lambda_H[B(1-u)H - \delta H], \tag{3}$$

where λ_K and λ_H are co-state variables. The first-order conditions are

$$e^{-\rho t}/C = \lambda_K, \tag{4}$$

$$\lambda_K A K^\alpha H^{1-\alpha}(1-\alpha)u^{-\alpha} = \lambda_H B H, \tag{5}$$

$$\dot{\lambda}_K = -\lambda_K[A\alpha K^{\alpha-1}(uH)^{1-\alpha} - \delta], \tag{6}$$

$$\dot{\lambda}_H = \lambda_H[\delta - B(1-u)] - \lambda_K A u^{1-\alpha}(1-\alpha)(K/H)^\alpha. \tag{7}$$

The transversality conditions are

$$\lim_{t\to\infty} \lambda_K K = 0, \tag{8}$$

$$\lim_{t\to\infty} \lambda_H H = 0. \tag{9}$$

It is convenient to introduce intensive variables $x = H/K$ and $y = C/K$. The equations of motion for intensive variables are:

$$\frac{\dot{x}}{x} = B - Bu + y - Au^{1-\alpha}x^{1-\alpha}, \tag{10}$$

$$\frac{\dot{y}}{y} = y - \rho + (\alpha - 1)Au^{1-\alpha}x^{1-\alpha}, \tag{11}$$

$$\frac{\dot{u}}{u} = -y + B/\alpha - B + Bu. \tag{12}$$

Intensive variables are constant in the steady state:

$$x^* = \left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} \frac{B}{\rho}, \tag{13}$$

$$y^* = \rho + \frac{1-\alpha}{\alpha} B, \tag{14}$$

$$u^* = \frac{\rho}{B}. \tag{15}$$

Extensive variables grow at the rate of $g^* = B - \rho - \delta$ in the steady state. The transversality conditions require $\rho > 0$. The equations of motion for intensive variables can be log-linearized around the steady state:

$$\begin{pmatrix} d\ln(x/x^*)/dt \\ d\ln(y/y^*)/dt \\ d\ln(u/u^*)/dt \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & -a_2 \\ a_3 & a_2 & a_3 \\ 0 & -a_2 & a_4 \end{pmatrix} \begin{pmatrix} \ln(x/x^*) \\ \ln(y/y^*) \\ \ln(u/u^*) \end{pmatrix}, \tag{16}$$

where $a_1 = -\frac{1-\alpha}{\alpha}B$, $a_2 = \rho + \frac{1-\alpha}{\alpha}B = y^*$, $a_3 = -\frac{(1-\alpha)^2 B}{\alpha}$, and $a_4 = \rho$. Parameters a_1, a_2, and a_4 are the eigenvalues of the Jacobian matrix. The speed of convergence equals $|a_1| = \frac{1-\alpha}{\alpha}B$. This speed corresponds to the result of Ortigueira and Santos (1997), who operated in a more general framework. The solution to the log-linearized Uzawa-Lucas model is given by the initial condition $[x(t=0) = x_0]$ and by the components of the eigenvector corresponding to the negative eigenvalue a_1:

$$\ln(x/x^*) = \ln(x_0/x^*)e^{-\frac{1-\alpha}{\alpha}Bt}, \tag{17}$$

$$\ln(y/y^*) = \ln(u/u^*) = \frac{a_3}{a_1 - a_2 - a_3}\ln(x/x^*). \tag{18}$$

Since $\frac{a_3}{a_1-a_2-a_3} > 0$, a high ratio of human to physical capital (x) is connected with high u and high $y = C/K$. This observation corresponds to one of the results in Mulligan and Sala-i-Martin (1993). One could easily derive that high x is associated with low C/H.

3 The imbalance effect

An important question is how the growth of output, Y, depends on x. Narrow output is defined by

$$Y = AK^\alpha (uH)^{1-\alpha} = A(H/K)^{-\alpha}u^{1-\alpha}H. \tag{19}$$

If $x = H/K$ is high, H/K decreases (which increases the growth of Y), u decreases, and H grows slowly (since u is high). If the capital share α is sufficiently low, the effects of u and H are more important than the effect of H/K. In this case we have a reverse (empirically implausible) imbalance effect between human and physical capital. To quantify this possibility, I derived a formula for the output growth, g_Y:

$$g_Y = -\alpha\frac{\dot{x}}{x} + (1-\alpha)\frac{\dot{u}}{u} + B(1-u) - \delta, \tag{20}$$

$$g_Y = \alpha A u^{1-\alpha} x^{1-\alpha} + B/\alpha - B - y - \delta. \tag{21}$$

After a log-linearization around the steady state, we have

$$g_Y = g^* + \left[\rho(1-\alpha)B\frac{1-2\alpha}{\alpha} + \frac{(1-\alpha)^2 B^2}{\alpha}\frac{1-3\alpha}{\alpha}\right]/(a_1 - a_2 - a_3)\ln(x/x^*). \tag{22}$$

Since $a_1 - a_2 - a_3 < 0$, the imbalance effect is empirically implausible if the capital share α is low (such as $\alpha = 1/3$). Mulligan and Sala-i-Martin (1993) discuss empirical evidence for a positive dependence of the output growth on x. I provide additional evidence in Duczynski (2003). Mulligan and Sala-i-Martin get a plausible imbalance effect (in some neighborhood of the steady state) in the Uzawa-Lucas model since they consider a relatively large capital share $\alpha = 0.5$ (they also consider the inverse elasticity of intertemporal substitution, θ, equal to 2; logarithmic preferences in the present model correspond to $\theta = 1$).

Mulligan and Sala-i-Martin (1993) observe that the growth of broadly defined output depends uniformly positively (i.e., not only in the neighborhood of the steady state) on the ratio of human to physical capital. Broadly defined output, Ω, includes investment in human capital measured in corresponding units of physical capital:

$$\Omega = Y + \frac{\lambda_H}{\lambda_K}B(1-u)H \tag{23}$$

Consistently with Mulligan and Sala-i-Martin's observation, we can derive that the imbalance effect for Ω is more realistic than for Y:

$$\frac{\lambda_H}{\lambda_K}B(1-u)H = AK^\alpha H^{1-\alpha}(1-\alpha)u^{-\alpha} - (1-\alpha)Y = (1-\alpha)Q - (1-\alpha)Y, \tag{24}$$

where $Q = AK^\alpha H^{1-\alpha}u^{-\alpha}$. Ω is a weighted average of Y and Q. We can show that the imbalance effect is plausible for Q:

$$g_Q = -\alpha\frac{\dot{x}}{x} - \alpha\frac{\dot{u}}{u} + B(1-u) - \delta, \tag{25}$$

$$g_Q = \alpha A u^{1-\alpha} x^{1-\alpha} - Bu - \delta. \tag{26}$$

After a log-linearization around the steady state, we obtain

$$g_Q = g^* + \left[\rho(1-\alpha)B\frac{1-2\alpha}{\alpha} - 2\frac{(1-\alpha)^2}{\alpha}B^2\right]/(a_1 - a_2 - a_3)\ln(x/x^*). \tag{27}$$

The imbalance effect is empirically plausible since $B > \rho$. Therefore, the overall imbalance effect for Ω is improved.

Measured output, Ω', is a weighted average of narrow output Y and broad output Ω. Mulligan and Sala-i-Martin (1993, p. 763) assume that measured output may include 25% of the education sector (wages of professors are counted, while foregone wages of students are not included). Therefore,

$$\Omega' = Y + 0.25\frac{\lambda_H}{\lambda_K}B(1-u)H. \tag{28}$$

In the Appendix we provide an extension of the present framework. We examine the imbalance effect for Y, Ω', and Ω for some preference specifications. The imbalance effect for Ω' tends to be implausible if, for example, $\alpha = 1/3$ and $\theta = 2$. This is an important result since this specification is quite realistic and Ω' is actually measured. We would obtain the same result for some neighborhood of this specification (definitely if α is lower and θ higher, or if α is slightly higher and θ slightly lower). Thus, there exists a relatively broad range of reasonable parameter specifications for which the Uzawa-Lucas model is inconsistent with empirical findings. On the other hand, the imbalance effect for Ω is realistic even if $\alpha = 0.25$ and $\theta = 3$. Generally there exists a tendency of the imbalance effect to be less plausible if α is low and θ is high.

4 Conclusion

For realistic parameter specifications, the Uzawa-Lucas model is believed to yield a positive dependence of the output growth on the ratio of human to physical capital (an empirically relevant imbalance effect). This paper shows that this result critically depends on the physical capital share in the physical capital sector. For a narrow concept of output and the physical capital share of 1/3, the imbalance effect tends to be empirically implausible. The imbalance effect is improved for a broad concept of output. Measured output is a weighted average of narrow and broad outputs. There exists a whole range of reasonable parameter specifications for which the Uzawa-Lucas model is inconsistent with empirical observations for measured output, at least in the neighborhood of the steady state. We believe that the present paper has pointed to an important weakness of the influential Uzawa-Lucas model.

Appendix

This Appendix extends the analysis in two ways. First, the instantaneous utility function exhibits a constant elasticity of intertemporal substitution,

$$U(C) = \frac{C^{1-\theta} - 1}{1-\theta}, \tag{29}$$

where θ is the inverse elasticity of intertemporal substitution. Second, we consider generally different depreciation rates for human and physical capital (δ_H and δ_K). The steady state is characterized by

$$x^* = \left(\frac{B + \delta_K - \delta_H}{\alpha A}\right)^{\frac{1}{1-\alpha}} \frac{B}{B - \delta_H + (\rho + \delta_H - B)/\theta}, \tag{30}$$

$$y^* = \frac{B + \delta_K - \delta_H}{\alpha} - \delta_K + \frac{\delta_H + \rho - B}{\theta}, \tag{31}$$

$$u^* = \frac{B - \delta_H + (\delta_H + \rho - B)/\theta}{B}. \tag{32}$$

These equations generalize (13)-(15), in which $\theta = 1$ and $\delta_K = \delta_H$. The steady-state growth rate of the economy is

$$g^* = \frac{B - \delta_H - \rho}{\theta}. \tag{33}$$

The Jacobian matrix of the system is

$$\begin{pmatrix} b_1 & b_2 & -b_2 \\ b_3 & b_2 & b_3 \\ 0 & -b_2 & b_4 \end{pmatrix},$$

where

$$b_1 = -\frac{1-\alpha}{\alpha}(B + \delta_K - \delta_H) < 0, \tag{34}$$

$$b_2 = \frac{B + \delta_K - \delta_H}{\alpha} - \delta_K + \frac{\delta_H + \rho - B}{\theta} = y^* > 0, \tag{35}$$

$$b_3 = \left(\frac{\alpha}{\theta} - 1\right)(1-\alpha)\frac{B + \delta_K - \delta_H}{\alpha}, \tag{36}$$

$$b_4 = \frac{\rho + \delta_H - B}{\theta} + B - \delta_H > 0. \quad (37)$$

We follow Mulligan and Sala-i-Martin (1993, p. 761) in assuming that $\theta > \alpha$ (very low values of θ are not observed empirically). Consequently, $b_3 < 0$. Parameters b_1, b_2, and b_4 are the eigenvalues of the Jacobian matrix. The speed of convergence is $|b_1| = \frac{1-\alpha}{\alpha}(B + \delta_K - \delta_H)$. Ortigueira and Santos (1997) find the same convergence coefficient. The speed of convergence is independent of the preference parameters θ and ρ. The solution of the log-linearized model is given by

$$\ln(x/x^*) = \ln(x_0/x^*)e^{-\frac{1-\alpha}{\alpha}(B+\delta_K-\delta_H)t}, \quad (38)$$

$$\ln(y/y^*) = \ln(u/u^*) = \frac{b_3}{b_1 - b_2 - b_3}\ln(x/x^*). \quad (39)$$

The transversality conditions are equivalent to

$$\frac{B - \delta_H - \rho}{\theta} < B - \delta_H. \quad (40)$$

This condition implies that $b_1 - b_2 - b_3 < 0$. Consequently, $\frac{b_3}{b_1-b_2-b_3} > 0$ for $\theta > \alpha$.

$$g_Y = -\alpha\frac{\dot{x}}{x} + (1-\alpha)\frac{\dot{u}}{u} + B(1-u) - \delta_H, \quad (41)$$

$$g_Y = \alpha A u^{1-\alpha}x^{1-\alpha} + B/\alpha - B - y + \delta_H - 2\delta_K + \frac{\delta_K - \delta_H}{\alpha}. \quad (42)$$

After a log-linearization around the steady state and some algebra, we obtain

$$g_Y = g^* + \frac{(1-\alpha)(B+\delta_K-\delta_H)}{b_1 - b_2 - b_3}(\psi_1 + \psi_2 + \psi_3)\ln(x/x^*), \quad (43)$$

where

$$\psi_1 = \left(B - \delta_H - \rho - \frac{2B + \delta_K - 2\delta_H - \rho}{\alpha} + \delta_K + \frac{B - \delta_H - \rho}{\theta}\right)/\theta, \quad (44)$$

$$\psi_2 = \frac{(B + \delta_K - \delta_H)(-2 + \alpha + 1/\alpha)}{\alpha}, \quad (45)$$

$$\psi_3 = \frac{\delta_K(\alpha - 1)}{\alpha}. \quad (46)$$

Mulligan and Sala-i-Martin (1993, p. 761) consider the following baseline specification: $\theta = 2$, $\rho = 0.04$, $\alpha = 0.5$, $\delta_H = \delta_K = 0.05$, and $B = 0.12$. For this specification, $\psi_1 + \psi_2 + \psi_3 < 0$, and g_Y really depends positively on x (since $b_1 - b_2 - b_3 < 0$). The imbalance effect becomes implausible for a higher θ ($\theta > 3$). Mulligan and Sala-i-Martin (1993, p. 763) get exactly the same result. As shown in the present paper, the imbalance effect for narrow output Y also becomes less realistic for a low α (for the above given baseline specification, the model is implausible if $\alpha < 0.5$ for $\theta = 3$, $\alpha < 0.449$ for $\theta = 2$, and $\alpha < 0.351$ for $\theta = 1$). We believe that these lower physical capital shares are relevant for a number of economies. For example, Mankiw (2000, p. 75) shows that the physical capital share in the United States has been roughly stable at 0.3 since the 1960s if we include depreciation in capital income and exclude proprietors' income from total income.

An important question is whether the Uzawa-Lucas model is consistent with empirical observations for measured output Ω' (as well as for broad output Ω). This depends on g_Q:

$$g_Q = g_Y - g_u, \tag{47}$$

$$g_Q = g^* + \frac{(1-\alpha)(B + \delta_K - \delta_H)}{b_1 - b_2 - b_3}(\psi_1 + \psi_2 + \psi_3 + \psi_4)\ln(x/x^*), \tag{48}$$

where

$$\psi_4 = \left(1 - \frac{\alpha}{\theta}\right)\frac{\alpha - 1}{\alpha^2}(B + \delta_K - \delta_H). \tag{49}$$

$\psi_4 < 0$ for $\theta > \alpha$, which improves the imbalance effect for Ω' or Ω. The behavior of the growth rates of Ω' and Ω depends on the weights of Y and Q:

$$\Omega = \alpha Y + (1-\alpha)Q, \tag{50}$$

$$\Omega' = [1 - 0.25(1-\alpha)]Y + 0.25(1-\alpha)Q \tag{51}$$

if we assume that 25% of the education sector is included in Ω'. The imbalance effect for Ω' or Ω depends not only on the weights of Y and Q, but also on the ratio of Q to Y. We are interested in the model's behavior in the neighborhood of the steady state. It holds that

$$\frac{Q^*}{Y^*} = \frac{1}{u^*} > 1. \tag{52}$$

For the baseline specification, $u^* = 1/3$ for $\theta = 1$, $u^* \doteq 0.46$ for $\theta = 2$, $u^* = 0.5$ for $\theta = 3$, and $u^* \doteq 0.52$ for $\theta = 4$. For a sufficiently low α, the dependence of g_Y on $\ln(x/x^*)$ has typically the opposite sign than the dependence of g_Q on $\ln(x/x^*)$. We compare the growth rate of Q (without the term g^*), multiplied by a corresponding weight and also by $\frac{Q^*}{Y^*}$, with the growth rate of Y (without the term g^*), multiplied by a corresponding weight. For other parameters given by the baseline specification, we get the following results: If $\alpha = 0.25$ and $\theta = 1$, the imbalance effect is implausible for narrow output and plausible for measured and broad outputs. However, for measured output the effect would become implausible if the portion of the education sector included in measured output were slightly less than 25%. If $\alpha = 0.25$ and $\theta = 2$ or $\theta = 3$, the effect is implausible for narrow and measured outputs, and plausible for broad output. If $\alpha = 1/3$ and $\theta = 1$, the imbalance effect is implausible for narrow output and plausible for measured and broad outputs. If $\alpha = 1/3$ and $\theta = 2$ or $\theta = 3$, the effect is implausible for narrow and measured outputs, and plausible for broad output. If $\alpha = 0.4$ and $\theta = 1$, the effect is plausible for all types of output. If $\alpha = 0.4$ and θ is 2 or 3, the effect is implausible for narrow output, and plausible for measured and broad outputs. However, if θ slightly exceeds 3 (while $\alpha = 0.4$), the effect for measured output becomes implausible.

In the end we should note that the physical capital share in the physical capital sector (α) should be calibrated for somewhat higher values than measured in national accounts because measured output includes a fraction of the education sector. My estimates show that for plausible parameter specifications, α should exceed the measured physical capital share by roughly 20% of the share. If we measure the share of 0.3 for the U.S. economy, then $\alpha = 0.35$ seems a realistic calibration. In this case the Uzawa-Lucas model is empirically implausible for a reasonably high θ.

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